MASTER'S THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Characterizing Universal Intervals in the Homomorphism Order of Digraphs

Author: Pablo Oviedo Timoneda

Advisor: Oriol Serra

Department: Departament de Matemàtiques, UPC

Academic year: 2019-2020

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

Characterizing the Universal Intervals in the Homomorphism Order of Digraphs.

Pablo Oviedo Timoneda

Advisor: Oriol Serra[1]

June, 2020

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Facultat de Matemàtiques i Estadística

[1]Departament de Matemàtiques, Universitat Politècnica de Catalunya

Abstract

In this thesis we characterize all intervals in the homomorphism order of digraphs in terms of universality. To do this, we first show that every interval of the class of digraphs containing cycles is universal. Then we focus our interest in the class of oriented trees (digraphs with no cycles). We give a density theorem for the class of oriented paths and a density theorem for the class of oriented trees, and we strengthen these results by characterizing all universal intervals in these classes. We conclude by summarising all statements and characterizing the universal intervals in the class of digraphs. This solves an open problem in the area.

Keywords: directed graph, digraph, homomorphism, partial order, homomorphism order, oriented path, oriented tree, density, universality.

Acknowledgements

First of all, I would like to express my deep gratitude to my supervisor, Prof. Oriol Serra, who made this work possible. His friendly guidance and expert advice have been essential throughout all stages of the work.

I also want to thank Prof. Jaroslav Nĕsetřil for his wise suggestions and support, it is always an honour to have helpful discussions with him.

My thanks are also extended to Prof. Jan Hubička for introducing me such amazing problems and encouraging me to try to solve them.

Finally, I would like to thank my family for being there and supporting me, specially in these uncertain times.

Contents

Introduction

The homomorphisms of graphs are the natural concept of morphisms in the category of graphs or directed graphs. As such they have been the object of intense study in Graph Theory, in Combinatorics in Logic and in Computer Science from different perspectives and motivations. Homomorphisms are natural generalizations of colorings, one of the central topics in Graph Theory. A whole area of problems in Computer Science, the constrained satisfaction problems, can be phrased in terms of homomorphisms. More recently, a large area of research has been devoted to the notion of graphs limits, which is based on graph homomorphisms. From the structural point of view, one can view the set of graph homomorphisms as a lattice which is surprisingly rich. This latter perspective is the central topic of this work.

A homomorphism from a directed graph G to a directed graph H is a mapping $f : V(G) \to V(H)$ which preserves adjacency and the direction of the arcs. We write $G \to H$ if there is an homomorphism from G to H and call two graphs equivalent if $G \to H$ and $H \to G$. A core is a minimal representative in an equivalence class under this relation. We say that $G \leq H$ if there is an homomorphism from G to H. The relation "$\leq$" induces a quasiorder on the class of digraphs. By selecting a core in each class of the above equivalence relation, this quasiorder can be turned into a partial order in the class of cores. The properties of this partial order are the main object of study in this work. It will be called the homomorphism order.

It turns out that the homomorphism order has a very rich structure, even when restricted to particular subclasses of graphs or of directed graphs. Nešetřil and Zhu [13] proved that even the restriction of the homomorphism order to one of the simplest classes of directed graphs, the class of oriented paths, is already rich enough as to represent every countable two dimensional partially ordered set (poset). The property is formulated in terms of the *density* of the partial order of oriented paths. This means that, for every interval $[P_1, P_2]$ in this partial order of oriented paths (with a set of well classified exceptions) there is a path P such that $P_1 < P < P_2$. In other words, every interval in the order, but for some exceptions, is dense. Fiala, Hubička, Long and Nešetřil [2] proved in 2017 that the homomorphism order in the class of (undirected) graphs has the fractal property in addition to the density property. This means that every interval $[G_1, G_2]$ in this order which is not a gap is universal in that it contains every countable partial order. This again illustrates the extremely rich structure of the homomorphism order.

In this context an open problem was formulated concerning the class of directed graphs. The problem asks for the characterization of gaps in this order, namely, pairs of digraphs G_1, G_2 such that $G_1 < G_2$ (and $G_2 \not\to G_1$) and there is no G in the class satisfying $G_1 < G < G_2$. Moreover, the problem asks for the universality of the intervals which are not gaps. The author initiated the investigation on this problem in his Bachelor thesis entitled “Universal intervals in the homomorphism order of digraphs” [14], which was developed in the department of Applied Mathematics of Charles University in Prague under the supervision of Prof. Jan Hubička. This thesis gives a complete answer to the problem by completing the project. The main results in this thesis are the following two theorems.

Theorem (3.2.1)**.** *Let T_1 be a tree and P_2 a path such that $T_1 < P_2$. If the height of P_2 is greater or equal to 4, then there exists a tree T satisfying $T_1 < T < P_2$.*

Theorem (3.3.4)**.** *Let P_1 and P_2 be two paths such that $P_1 < P_2$. If the height of P_2 is greater or equal to 4, then the interval $[P_1, P_2]$ is universal.*

Theorem 3.2.1 is a generalisation of the density theorem for paths proved in [13] by Nešetřil and Zhu. Theorem 3.3.4 strengthens Theorem 3.2.1 by showing that every interval of the class of oriented paths of height greater or equal to 4, in addition to be dense, is universal. The two theorems provide a complete answer to the open questions on gaps, density and universality of the homomorphism order in the class of directed graphs which remained open.

The thesis is organised as follows. In Chapter 1 we present the basic definitions and properties of homomorphisms and digraphs, and we introduce the homomorphism order which is the main focus of this study . The content of the chapter is based in the monography *Graphs and homomorphisms* by Pavol Hell and Jaroslav Nešetřil [5] which is the standard reference on the topic.

In the first sections of Chapter 2 we characterize dense intervals and gaps in the homomorphism order of digraphs. Then, in the second section of Chapter 2, we define the concepts of universality and the fractal property and we show that intervals of the form $[G, H]$ where the core of H contains a cycle are universal.

In Chapter 3 we focus our interest on the class of oriented trees. First, we introduce the basic definitions and properties of paths and trees. Then, we split the study into the class of paths, and the class of proper trees (trees which are not paths). For each of these classes we give a density theorem and characterize its universal intervals. These are the main results in the thesis, collected, in order of appearance, in theorems 3.2.1, 3.3.4, 3.4.2 and 3.5.2.

Finally, in Chapter 4 we summarise the results obtained and state our desired result: the characterization of all universal intervals in the homomorphism order of digraphs. We conclude by identifying some lines of further research and discuss some open problems. In particular we discuss the extension of the results to the general class of relational structures, a concept which generalises the notion of a directed graph.

The thesis is meant to be mainly self contained. However, we shall state some known theorems without proof when their difficulty and their not so close relation to the main results make it reasonable to omit them.

The preliminary results of the thesis were presented in the last edition of the *European Conference on Graph Theory, Combinatorics and Applications* held in Bratislava in August 2019, and the full solution to the problem has been selected for publication in the special issue of the *European Journal of Combinatorics* devoted to the conference.

Chapter 1

Graph Homomorphisms and the Homomorphism Order

1.1 Digraphs and Homomorphisms

A *digraph* G is an ordered pair of sets (V, A) where $V = V(G)$ is a set of elements called *vertices* and $A = A(G)$ is a binary relation on V. The elements (u, v) of $A(G)$ are called *arcs* and we shall denote them as uv. An arc of the form (u, u) is called a *loop*. A digraph G is *finite* if $V(G)$ is finite. Note that in this case $A(G)$ is also finite. We say that a digraph G is *symmetric*, or *irreflexive*, or etc., if $A(G)$ is symmetric, or irreflexive, or etc., respectively. Note that a digraph is irreflexive if and only if it contains no loops.

A *simple graph* or *graph* G is an ordered pair of sets (V, E) where $V = V(G)$ is a set of *vertices* and $E = E(G)$ is a set of *edges*, which are sets of vertices of size two. A graph G is *finite* if $V(G)$ is finite. Most commonly, in texts on graph theory, *graph* means "finite simple graph".

In this thesis we shall assume graphs to be finite and simple, and digraphs to be finite and irreflexive. In Chapters 2 and 3 we may write, by some abuse of notation, G instead of $V(G)$ to denote the vertex set of a digraph G.

Observe that one can also define graphs as symmetric digraphs by replacing each edge $\{u, v\}$ by the two arcs (u, v) and (v, u). Thus, every definition or property on digraphs can be applied to graphs. In fact, we shall view the class of graphs as a subclass of the class of digraphs via their corresponding symmetric digraphs.

An *orientation of* a graph G is a digraph obtained by replacing each edge $\{u, v\}$ by exactly one of the arcs (u, v) or (v, u). An *oriented graph* is a digraph obtained from an orientation of some graph. It can be observed that a digraph is an oriented graph if and only if it has no pair of symmetric arcs.

For the rest of the thesis we will be dealing almost always with digraphs. However, most of the definitions and properties can also be applied to graphs by considering a graph as a symmetric digraph. We will write arcs (u, v) simply as uv.

Given a digraph G, if $uv \in A(G)$ we shall say that v is an *outneighbour* of u and u is an *inneighbour* of v. We shall say that u and v are adjacent as long as at least one of uv, vu is an arc of $A(G)$; in this case we shall also say that u and v are *neighbours*. The number of neighbours of a vertex u is called the *degree* of u and is denoted $d(u)$, while $d^+(u)$ denote the number of outneighbours and $d^-(u)$ the number of inneighbours. Given an arc $uv \in A(G)$ we say that uv is *incident* to u and v.

Given two digraphs G and H, we say that G is a *subgraph* of H if $V(G) \subseteq V(H)$ and $A(G) \subseteq A(H)$. In such case we write $G \subseteq H$. Given a digraph H and a subset $V(G) \subseteq V(H)$, the *digraph induced* by $V(G)$ is the digraph $G = (V(G), A(G))$ where $A(G) = \{uv \mid u, v \in V(G)\ uv \in A(H)\}$. In this case we say that G is an *induced subgraph* of H. Given two digraphs G, H such that $G \subseteq H$, then $H \backslash G$ is the digraph with $V(H) \backslash V(G)$ as set of vertices and $A(H) \backslash A(G)$ as set of arcs.

Finally, a digraph is *complete* if every pair of vertices are adjacent. We shall denote by K_n the complete graph with n vertices. We shall refer to an arbitrary orientation of K_n as $\vec{K}_n$. Note that $\vec{K}_1 = K_1$.

Let G and H be two digraphs. A *homomorphism* from G to H is a mapping $f : V(G) \to V(H)$ such that if uv is an arc in G then $f(u)f(v)$ is an arc in H; in other words, $uv \in A(G)$ implies $f(u)f(v) \in A(H)$. Note that homomorphisms preserve not just adjacency, but also the direction of arcs. Thus, a homomorphism $f : G \to H$ induces a map $f : A(G) \to A(H)$ defined as $f(uv) = f(u)f(v)$. By some abuse of notation we still denote by f this induced mapping between the sets of arcs. If there exists a homomorphism from G to H we shall write $G \to H$ and we shall say that G is *homomorphic to* H. If there is no such homomorphism we shall write $G \nrightarrow H$. It is easy to check that the composition $f \circ g$ of homomorphisms $g : G \to H$ and $f : H \to X$ is a homomorphism from G to X.

Let G and H be two digraphs and $f : G \to H$ a homomorphism. The *image* of f, denoted $\mathrm{Im}(f)$ or $f(G)$, is the subgraph of H induced by the vertices $\{v \in V(H) \mid \exists u \in V(G) \text{ s.t. } f(u) = v\}$. The *preimage* of a vertex $v \in V(f(G))$, denoted $f^{-1}(v)$, is the set $\{u \in V(G) \mid f(u) = v\}$. Observe that every two vertices in $f^{-1}(v)$ are non adjacent since otherwise vv would be a loop in $A(H)$. An *independent set* is a digraph S such that every two vertices in $V(S)$ are non adjacent. Thus, $f^{-1}(v)$ is an independent set.

An *isomorphism* from G to H is a bijective homomorphism $f : V(G) \to V(H)$

which also preserves non-adjacency. This means that a bijective mapping $f : V(G) \to V(H)$ is an isomorphism if $f(u)f(v) \in A(H)$ if and only if $uv \in A(G)$. In this case, the mapping $f^{-1} : V(H) \to V(G)$ is also a homomorphism. We shall denote it as $f^{-1} : H \to G$. The composition $f \circ f^{-1} : G \to G$ is the identity on the digraph G. If there exists an isomorphism from G to H we shall say that G and H are *isomorphic*. Note that if $f : G \to H$ is a injective homomorphism then G is isomorphic to $f(G)$.

An *endomorphism* of a digraph G is a homomorphism from G to itself. The set of all endomorphisms of a digraph G is denoted by $\mathrm{End}(G)$. An *automorphism* is an isomorphism from a digraph G to itself. The set of all automorphisms of a digraph G, denoted $\mathrm{Aut}(G)$, is a group under composition. We have that $\mathrm{Aut}(G) \subseteq \mathrm{End}(G)$, but $\mathrm{End}(G)$ is not necessarily a group. It can be checked that a bijective endomorphism is already an automorphism. For this reason endomorphisms differ from automorphisms in that their image can be a proper subgraph of G.

1.2 Some Properties of Homomorphisms

The fact that homomorphisms preserve adjacency and direction of arcs has interesting implications. One of the most direct implications is how paths and cycles behave under homomorphisms. So let us start this section with its definitions.

A *walk* in a digraph G is a sequence of vertices $v_0, v_1, \ldots, v_k \in V(G)$ together with a sequence of arcs $a_1, a_2, \ldots, a_k \in A(G)$ such that for each $i = 1, \ldots, k$, a_i is an arc incident to v_{i-1} and v_i. The arcs of the form $v_{i-1}v_i$ are called *forward arcs* and the arcs of the form v_iv_{i-1} are called *backwards arcs*. The integer k is called the *length* of the walk. The *net length* is the difference between the number of forward arcs and the number of backward arcs. In the case of a graph the net length is set to be zero. A walk is *closed* if $v_0 = v_k$.

A *path* is a walk in which vertices and arcs in the sequences are pairwise distinct. A *cycle* is a closed path. Since a path and a cycle are walks, the definitions of length and net length are also applicable.

A walk, path, etc., in which all arcs are forward arcs are called *directed* walk, *directed* path, etc., respectively. We shall denote by P_k, C_k, $\vec{P}_k$ and $\vec{C}_k$ the path, cycle, directed path and directed cycle of length k respectively. Note that P_k has $k+1$ vertices and k arcs while C_k has k vertices and k arcs.

Proposition 1.2.1. *Let G and H be digraphs and $f : G \to H$ a homomorphism. If $v_0, \ldots, v_k$ and $a_1, \ldots, a_k$ is a walk in G then $f(v_0), \ldots f(v_k)$ and $f(a_1), \ldots, f(a_k)$ is* *a walk in H of the same length and net length.*

Proof. It is clear since homomorphism preserves adjacency and direction of arcs. $\square$

The same argument can be applied for particular cases of walks.

Corollary 1.2.2. *Let G be a digraph.*

- *If $f : P_k \to G$ is a homomorphism, then $f(P_k)$ is a walk in G.*
- *If $f : C_k \to G$ is a homomorphism, then $f(C_k)$ is a closed walk in G.*
- *If $f : \vec{P}_k \to G$ is a homomorphism, then $f(\vec{P}_k)$ is a directed walk in G.*
- *If $f : \vec{C}_k \to G$ is a homomorphism, then $f(\vec{C}_k)$ is a directed closed walk in G.*

In all the cases the length and net length is preserved.

A digraph is *connected* if every two vertices are joined by a path. A *component* or *connected component* of a digraph G is a maximal connected subgraph. The distance $d_G(u, v)$ between two vertices u, v in a digraph G is the length of a shortest path joining them. We set $d_G(u, v) = \infty$ if there is no path joining u and v. We emphasize that the above definitions refer to the underlying graph obtained from G by ignoring the orientations of the arcs.

Corollary 1.2.3. *Let $f : G \to H$ be a homomorphism where G is connected. Then $d_H(f(u), f(v)) \leq d_G(u, v)$ for any $u, v \in G$.*

Proof. Let $u = u_0, \dots, u_k = v$ be the sequence of vertices of a path of length k in G. Since the image of a path of length k is a walk of the same length and every walk from $f(u)$ to $f(v)$ contains a path from $f(u)$ to $f(v)$, it follows that $d_H(f(u), f(v)) \leq k$. $\square$

We shall use this simple result several times in this thesis.

Colouring is one of the most studied concepts in graph theory. There is an striking amount of problems and applications related to it. Moreover, colourings are really related to homomorphisms. In fact, homomorphisms can be viewed as a generalitzation of colourings. Let us see why.

A graph G is *k-colorable* if there exists a partition of $V(G)$ into k independent sets. Such a partition is called a *k-colouring* of G. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum k such that G is k-colorable.

Let G and H be two graphs. If there exists a homomorphism $f : G \to H$ it is often said that G is *H-colorable* or that f is a *H-colouring* of G. The reason to this is that if we have that $G \to H$ via a homomorphism f, then for each $v \in H$ if we take the independent set $f^{-1}(v)$ we obtain a partition of $V(G)$ into k independent

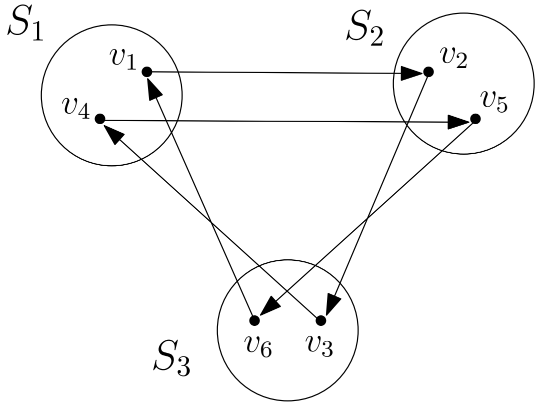

Figure 1.1: A $\vec{C}_3$-colouring of $\vec{C}_6$.

sets, $k \leq |V(H)|$. This in fact, is the condition of a graph for being k-colorable. Then we can say that if $G \to H$ and $n = |V(H)|$ then G is n-colorable. From these facts we can deduce the following proposition.

Proposition 1.2.4. *A graph G is n-colorable if and only if $G \to K_n$. Moreover, the homomorphisms from G to K_n are precisely the n-colourings of G.*

Let H be a graph and let $n = |V(H)|$. We have that the condition of being homomorphic to H is stronger than being n-colorable. If a graph G is homomorphic to H we know that we can make a partition of $V(G)$ into n independent sets but, moreover, this partition might have some restrictions involving the non-adjacency of the vertices from two different independent sets. To think of homomorphisms as a generalisation of colourings can be very useful to understand better their behaviour and properties.

Corollary 1.2.5. *If $G \to H$ then $\chi(G) \leq \chi(H)$.*

Proof. Let $n = \chi(H)$. We know that $G \to H \to K_n$, then $G \to K_n$ which implies that $\chi(G) \leq n$. $\square$

There is a similar result concerning the odd girth. The *girth* of a digraph is the minimum length of a cycle in it. Similarly, the *odd girth* of a non bipartite graph is the minimum length of an odd cycle in it. It is known from Theorem 1.2.10 below that the property of a graph of being non bipartite is equivalent to contain at least one odd cycle.

Proposition 1.2.6. *Let G and H be two non bipartite graphs. If $G \to H$ then the odd girth of G is greater or equal to the odd girth of H.*

Proof. Let $v_0, \ldots, v_k = v_0$ be the sequence of vertices of an odd cycle in G of minimum length k. Let $f : G \to H$ be a homomorphism. Then $f(v_0), \ldots, f(v_k) =$

$f(v_0)$ is a closed walk of length k in H. Since we can not obtain an odd number from the sum of even numbers, there exists at least one odd cycle in the sequence $f(v_0), \ldots, f(v_k) = f(v_0)$ of length less or equal to k. $\square$

In Proposition 1.2.4 we have defined colourability in terms of homomorphisms. But this is not the only property of graph theory that can be expressed in such way. We say that a digraph G is *balanced* if every cycle in G has net length equal to zero. We denote by $\vec{T}_k$ the digraph with vertices $v_0, \ldots, v_k$ and arcs $v_i v_j$ for every $i < j$. $\vec{T}_k$ is an orientation of the complete graph K_k called the transitive tournament.

Proposition 1.2.7. *A connected digraph G with n vertices does not contain a directed cycle if and only if $G \to \vec{T}_{n-1}$*

Proof. It is easy to check that $\vec{T}_{n-1}$ does not contain a directed cycle. Suppose now that G contains a directed cycle $\vec{C} \subseteq G$. If $f : G \to \vec{T}_{n-1}$ is a homomorphism then we have that $f(\vec{C})$ is a directed cycle in $\vec{T}_{n-1}$ which is a contradiction.

Let G be a digraph of n vertices with no directed cycles. We shall now label each vertex v by the maximum number of arcs in a directed walk that ends in it. Since G is free of directed cycles, it is easy to check that this labelling is injective and labels the vertices from 0 to n-1. Finally, this labelling induces a homomorphism from G to $\vec{T}_{n-1}$ by mapping each vertex with label i to the vertex $v_i \in \vec{T}_{n-1}$. $\square$

Proposition 1.2.8. *A digraph G with n vertices is balanced if and only if $G \to \vec{P}_{n-1}$.*

Proof. Note that $\vec{P}_{n-1}$ is balanced as it has no cycles. Suppose G is a non balanced digraph so there exists some cycle C in G with net length different from zero. If $f : G \to \vec{P}_{n-1}$ is a homomorphism then we have that $f(C)$ is a cycle in $\vec{P}_{n-1}$ with net length different from zero which is a contradiction since $\vec{P}_{n-1}$ is balanced.

Let G be a balanced graph with n vertices. We shall label its vertices by integers as follows. In each component of G pick one arbitrary vertex and label it to 0. Once a vertex has been labelled by the integer i, label all of its outneighbours by $i + 1$ and all of its inneighbours by $i - 1$. It is easy to check that these procedure will give every vertex a unique label since G is balanced. Once every vertex is labelled, we can shift the labels so that the smallest one starts with 0. Note that since G has n vertices the maximum label of a vertex will be at most $n - 1$. This final labelling induces a homomorphism form G to $\vec{P}_{n-1}$ by mapping each vertex with label i to the vertex $v_i \in \vec{P}_{n-1}$. $\square$

In the previous proof we have assigned a labelling to each vertex of a connected component of a balanced digraph. So given a connected balanced digraph G the previous labelling is unique and assigns each vertex an integer. We call the label of a vertex v the *level* of v, and we call the maximum level of a vertex in G the *height* of G.

Corollary 1.2.9. *If G and H are two balanced digraphs of the same height, then any homomorphism from G to H preserves the levels of vertices.*

Proof. As we have seen in the proof of Proposition 1.2.8, if G is a digraph of height k then there is a unique homomorphism from G to $\vec{P}_k$ which is the one that maps each vertex with level i to the vertex $v_i \in \vec{P}_k$. Suppose that there exists some homomorphism $f : G \to H$ which does not preserve the level of some vertex and let $g : H \to \vec{P}_k$ be a homomorphism. We know that g preserves the level of vertices. Then the composition $g \circ f : G \to H \to \vec{P}_k$ is a homomorphism from G to $\vec{P}_k$ which does not preserve the level of some vertex, which is a contradiction. □

There exists many cases in which the existence of some homomorphisms is equivalent to the non existence of some other homomorphisms. These are called *homomorphism dualities*. There is one simple example of these dualities applied to graphs and it follows from the well known theorem of König, which states that a graph is *bipartite*, which means 2-colorable, if and only if it has not odd cycles. This theorem can be expressed in terms of a homomorphism duality.

Theorem 1.2.10 (König's theorem)**.** *A graph G satisfies $G \to K_2$ if and only if $C_k \nrightarrow G$ for every odd integer $k \geq 3$.*

There is also another simple example of a homomorphism duality, in this case applied to digraphs. The following proposition was shown in [11] by Nešetřil and Pultr.

Proposition 1.2.11. *A digraph G satisfies $G \nrightarrow \vec{T}_{k-1}$ if and only if $\vec{P}_k \to G$.*

Proof. The longest directed path in $\vec{T}_{k-1}$ has length $k-1$ while $\vec{P}_k$ is a directed path of length k, so $\vec{P}_k \nrightarrow \vec{T}_{k-1}$. It follows that if $\vec{P}_k \to G$ then $G \nrightarrow \vec{T}_{k-1}$.

Suppose that $\vec{P}_k \nrightarrow G$. Then the labelling of the proof of Proposition 1.2.7 is well defined, since G has not directed paths of length greater or equal to k and it labels the vertices of G from 0 to $k-1$. Thus the labelling induces a homomorphism $G \to \vec{T}_{k-1}$. □

This proposition implies the following well known fact that relates graphs and digraphs.

Corollary 1.2.12. *A graph G is k-colorable if and only if there exists an orientation of G which does not contain the directed path $\vec{P}_k$.*

Proof. If G is k-colorable we can make a partition of $V(G)$ into k independent sets $V_1, \dots, V_k$. We shall replace each edge in G by an arc as follows. Consider an edge uv in G. We know that $u \in V_i$ and $v \in V_j$ for some $0 \leq i, j \leq k$ such that $i \neq j$. We shall replace the edge uv by the arc uv if $i < j$ and by the arc vu otherwise. It is clear that this replacement gives us an orientation of G homomorphic to $\vec{T}_{k-1}$. Then Proposition 1.2.11 implies that $\vec{P}_k \nrightarrow G$.

On the other hand, Let $\vec{G}$ be an orientation of G which does not contain the directed path $\vec{P}_k$. We know by Proposition 1.2.11 that there exists a homomorphism $f : \vec{G} \to \vec{T}_{k-1}$. Then the sets $f^{-1}(v_0), \ldots, f^{-1}(v_{k-1}) \subseteq V(\vec{G}) = V(G)$, where $v_0, \ldots, v_{k-1}$ are the vertices of $\vec{T}_{k-1}$, are a k-colouring of G. □

Given two digraphs G and H, the *disjoint union* or *sum* of G and H is the digraph $G+H$ which has the vertex set $V(G+H) = V(G) \sqcup V(H)$ and arcs $uv \in A(G+H)$ if $uv \in A(G)$ or $uv \in A(H)$. The same definition is applied to graphs. Note that the sum of two graphs is also a graph. The sum of digraphs has simple and interesting properties.

Proposition 1.2.13. *A digraph G is not connected if and only if G is equal to the sum of two digraphs.*

Proof. It is clear from the definition of sum that the sum of two digraphs is not connected. On the other hand, if G is not connected it has at least two components. Let G_1 be equal to one connected component and let G_2 be equal to $G \backslash G_1$. It is easy to check that $G = G_1 + G_2$. □

More related to homomorphisms are the following properties.

Proposition 1.2.14. *Let G, H and X be digraphs.*

- $G \to G + H$.
- *If $G \to X$ and $H \to X$ then $G + H \to X$.*

Proof. It follows from the definition of $G + H$ that the inclusion $i : G \to G + H$ is a homomorphism.

Moreover, if $f_G : G \to X$ and $f_H : H \to X$ are homomorphisms, then it is easy to check that the mapping $f : G + H \to X$ defined as $f(u) = f_G(u)$ for all $u \in G$ and $f(v) = f_H(v)$ for all $v \in H$ is also a homomorphism. □

Note that the homomorphism f defined in the previous proof satisfies $f_G = f \circ i_G$ and $f_H = f \circ i_H$, and it is the unique mapping which satisfies this property. In fact, this uniqueness property characterizes the sum of digraphs and inclusions.

Theorem 1.2.15 (Characterization of the Sum)**.** *For any digraphs G and H there exists a unique (up to isomorphism) digraph S and unique homomorphisms $s_G : G \to S$ and $s_H : H \to S$ such that for every digraph X to which G and H are homomorphic via $f_G : G \to X$ and $f_H : H \to X$, there exists a unique homomorphism $f : S \to X$ satisfying $f \circ s_G = f_G$ and $f \circ s_H = f_H$.*

Given two digraphs G and H, the *product* of G and H is the digraph $G \times H$ which has the vertex set $V(G \times H) = V(G) \times V(H)$ and arcs $(u,v)(u',v') \in A(G \times H)$ whenever uv and $u'v'$ are arcs in $A(G)$ and $A(H)$ respectively. See some examples in Figure 1.2. The same definition is applied to graphs. Note that the product of two graphs is also a graph. This is the product in the category of digraphs (or in the one of graphs) as it has the universal property in the category.

Proposition 1.2.16. *Let G, H and X be digraphs.*

- $G \times H \to G$ *and* $G \times H \to H$.
- *If* $X \to G$ *and* $X \to H$ *then* $X \to G \times H$.

Proof. Consider the two *projections* $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ defined as $\pi_1(u,v) = u$ and $\pi_2(u,v) = v$ for all $(u,v) \in V(G \times H)$. It follows from the definition of $G \times H$ that π_1 and π_2 are homomorphisms.

Moreover, if $f_1 : X \to G$ and $f_2 : X \to H$ are homomorphisms, then it is easy to check that the mapping $f : X \to G \times H$ defined as $f(x) = (f_1(x), f_2(x))$ is also a homomorphism. □

Corollary 1.2.17. *For digraphs G and H,* $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.

Proof. It easily follows from the fact that $G \times H \to G$ and $G \times H \to H$. □

Note that the homomorphism f defined in the proof of Proposition 1.2.16 satisfies $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$, and is the unique mapping which satisfies this property. In fact, as with the sum, this uniqueness characterize the product digraph and projections.

Theorem 1.2.18 (Characterization of the Product)**.** *For any digraphs G and H, there exists a unique (up to isomorphism) digraph P and unique homomorphisms $p_1 : P \to G$ and $p_2 : P \to H$ such that for every digraph X homomorphic to G and H via $f_1 : X \to G$ and $f_2 : X \to H$, there exists a unique homomorphism $f : X \to P$ satisfying $p_1 \circ f = f_1$ and $p_2 \circ f = f_2$.*

Theorem 1.2.15 and Theorem 1.2.18 allows us to define both, the sum and the product, in a more general way. Both operations can be defined as the unique digraph which satisfies the properties of its characterization theorem.

One might ask if both operations are commutative and associative. Moreover, if the product is distributive over the sum. Indeed, it is not difficult to see that the commutative, associative and distributive property holds for the sum and product of digraphs.

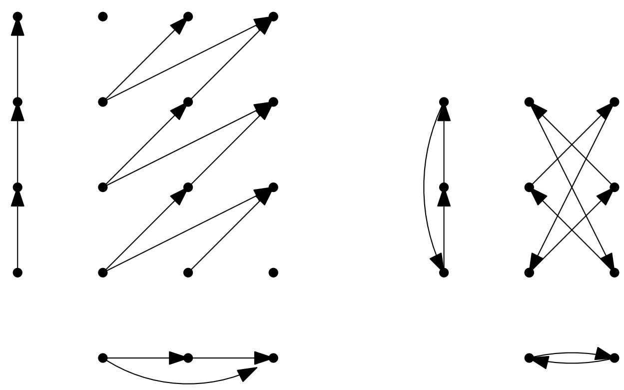

Figure 1.2: Product digraphs $\vec{P}_3 \times \vec{T}_2$ and $\vec{C}_3 \times K_2$.

1.3 Cores and Rigid Digraphs

A *retraction* of a digraph G is a homomorphism $r : G \to H \subseteq G$ which satisfies $r(x) = x$ for all vertices $x \in V(H)$. If H admits a retraction from G we shall say that H is a *retract of* G. Retractions are at the heart of the problem of extending homomorphisms. However, we are interested in them since they allow us to define cores, which are one of the fundamental concepts of this thesis.

Proposition 1.3.1. *Let G be a digraph and let H be a subgraph of G. Then H is a retract of G if and only if any homomorphism $f : H \to X$ can be extended to a homomorphism $F : G \to X$.*

Proof. Suppose that H is a retract of G and let $f : H \to X$ be a homomorphism. We know there exists a retraction $r : G \to H$ such that $f(v) = v$ for every $v \in V(H)$. It follows, then, that $F = (f \circ r) : G \to X$ is a extension of f.

Consider the identity mapping $id : H \to H$. Suppose that id can be extended to a homomorphism $F : G \to H$. Then F is a retraction and thus H is a retract of G. □

We may observe that the composition of retractions is also a retraction. This implies that if a digraph K is a retract of H and H is a retract of G, then K is a retract of G. Note also that if G retracts to a proper subgraph H, then H must have strictly less number of vertices than G. So there must exist some subgraph of G which does not admit a retraction. For this reason we shall define the following concept.

A *core* is a digraph which does not retract to a proper subgraph. Cores are a fundamental concept to well define the homomorphism order.

Proposition 1.3.2. *Every digraph contains a core.*

The following proposition allows us to think about cores forgetting the concept of retraction.

Proposition 1.3.3. *A digraph G is a core if and only if G is not homomorphic to a proper subgraph.*

Proof. It is clear that if G retracts to a proper subgraph, then it is homomorphic to it. Conversely, if G is homomorphic to a proper subgraph, let H be a proper subgraph of G with the fewest number of vertices to which G is homomorphic. Then H is not homomorphic to a proper subgraph of itself. So any homomorphism $H \to H$ is an automorphism. Consider a homomorphism $f : G \to H$ and let $h = f|_H : H \to H$ be the restriction of f to H. Since h is an automorphism there exists an inverse automorphism h^{-1}. Observe that $h^{-1} \circ f$ is a retraction of G to H, and hence G is not a core. □

Observe that in the last proof we have shown that if H is a core, then every homomorphism $H \to H$ is an automorphism. This observation is really important since we shall use it several times during this thesis. For this reason let us state it as a Corollary.

Corollary 1.3.4. *Every homomorphism from a core to itself is an automorphism.*

We say that two digraphs which are homomorphic to each other are *homomorphically equivalent*. It is easy to check that this relation is in fact an equivalence relation. We could maybe think that two different digraphs which are homomorphically equivalent must have the same amount of vertices or arcs. But this is not the case. However, there are a lot of properties that homomorphically equivalent digraphs will have in common. They are all properties related to homomorphisms. One example is the chromatic number. It follows from Corollary 1.2.5 that two graphs which are homomorphically equivalent have the same chromatic number. The same happens with the odd girth. But the property in which we are most interested is that homomorphically equivalent digraphs share the same core. This will allows us to split the set of all digraphs into equivalence classes via the homomorphic equivalence and choose for each class its correspondent core as its representative.

Proposition 1.3.5. *Every digraph is homomorphically equivalent (up to isomorphism) to a unique core .*

Proof. First of all, observe that every digraph is homomorphically equivalent to its core. Suppose now that H and H' are two different cores of a digraph G. From the transitive property of the equivalence relation H and H' are also homomorphically equivalent. Let $f : H \to H'$ and $g : H' \to H$ be homomorphisms. Since H and H' are cores, both $(f \circ g)$ and $(g \circ f)$ are automorphisms. Hence, H and H' are isomorphic. □

Corollary 1.3.6. *Two homomorphically equivalent digraphs have the same core (up to isomorphism).*

Proof. Let G and G' be two homomorphically equivalent digraphs and let H and H' be its core respectively. Observe that both G and G' are homomorphically equivalent to its respective cores. Since G and G' are also homomorphically equivalent, it follows from the transitive property that H and H' are homomorphically equivalent. Then, due to Proposition 1.3.5, H and H' are isomorphic. $\square$

With this results we know that every digraph in the same equivalence class has the same core. This is a very useful fact since we can generalise the results obtained for some core to all digraphs in its equivalence class. This is true since all digraphs in the same equivalence class have the same homomorphism properties. Some examples of cores are the complete graphs K_n. As a matter of fact, the set of all bipartite graphs is exactly the equivalence class which contains K_2 as its core. In digraphs, all directed paths $\vec{P}_k$ and directed cycles $\vec{C}_k$ are cores, as well as all digraphs $\vec{T}_n$. But there are plenty of more examples of cores. In fact, asymptotically almost all digraphs are cores (see e.g. [5]).

A digraph G is *rigid* if it is a core and the only automorphism $f : G \to G$ is the identity mapping. Recall that any homomorphism from a core to itself is an automorphism, so if G is rigid then the only homomorphism $f : G \to G$ is the identity mapping.

Recall from Proposition 1.3.2 that every digraph contains a core. Since the core is unique (up to isomorphism), it follows that the core is a subgraph of every graph in its equivalence class.

Proposition 1.3.7. *Let G be a digraph and let $C \subseteq G$ be a rigid digraph. Let $f : G \to C$ and $g : C \to G$ be homomorphisms. Then*

- *The restriction $f|_C$ is the identity mapping.*
- $g(C) \cong C$
- *For any vertex $v \in C$, $g(v)$ satisfies $f(g(v)) = v$.*

Proof. First observe that C is a core, so in particular, C is the core of G. Since C is rigid $f|_C : C \to C$ must be the identity mapping by definition. Consider the composition $(f \circ g) : C \to C$. It follows by definition that $(f \circ g)$ is also the identity mapping, then $g()$ $\square$

1.4 The Partial Order of Homomorphisms

A *partially ordered set* is a set $\mathcal{P}$ (not necessarily finite) together with a binary relation, usually denoted by $\leq$, satisfying the following properties:

- Reflexivity: $x \leq x$ for all $x \in \mathcal{P}$.
- Transitivity: $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in \mathcal{P}$.
- Antisymmetry: $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in \mathcal{P}$.

The relation $\leq$ is commonly referred to as a *partial order* on the set $\mathcal{P}$. For our purpose, since we are interested in the set of all finite digraphs, we shall only consider countable partially ordered sets.

Let $\vec{\mathscr{G}}$ be the set of all finite digraphs. Let us write $G \leq H$ for $G \to H$ (with $G, H \in \vec{\mathscr{G}}$). Observe that the relation $\leq$ ("being homomorphic to") is reflexive and transitive. We shall refer to this relation as the *homomorphism order*. However, it is not antisymmetric since homomorphically equivalent graphs might not be equal. A binary relation that is reflexive and transitive is called a *quasiorder*. Thus, the homomorphism order $\leq$ defines a quasiorder on $\vec{\mathscr{G}}$.

There are standard ways to transform a quasiorder into a partial order. One of them is by choosing a representative for each equivalence class. In our case we shall choose the cores to be the representative of each class as we have discussed in the previous section. Let $\vec{\mathscr{C}}$ be the set of all cores in $\vec{\mathscr{G}}$. Then the following theorem follows.

Theorem 1.4.1. $(\vec{\mathscr{C}}, \leq)$ *is a partially ordered set.*

In consequence, the homomorphism order is a partial order on $\vec{\mathscr{C}}$.

Let $\mathscr{G}$ be the set of all finite graphs and let $\mathscr{C}$ be the set of all cores in $\mathscr{G}$. Since we can view graphs as symmetric digraphs we have that $\mathscr{G} \subset \vec{\mathscr{G}}$. Note that the core of a symmetric digraph is also a symmetric digraph, then we also have that $\mathscr{C} \subset \vec{\mathscr{C}}$. It follows that $(\mathscr{C}, \leq)$ is a suborder of $(\vec{\mathscr{C}}, \leq)$.

Corollary 1.4.2. $(\mathscr{C}, \leq)$ *is a partially ordered set.*

The structure of the homomorphism order is rich in interesting properties that we shall discuss during this thesis, in particular in Chapter 2. But let us start with a simple one. A *lattice* is a partially ordered set in which every two elements have a least upper bound and a greatest lower bound.

Proposition 1.4.3. $(\vec{\mathscr{C}}, \leq)$ *is a lattice.*

Proof. Indeed, given two cores G and H, the least upper bound and greatest lower bound are the cores of the digraphs $G + H$ and $G \times H$ respectively. It follows from Proposition 1.2.14 that $G \leq G + H$, $H \leq G + H$, and if a core X satisfies $G \leq X$ and $H \leq X$ then $G + H \leq X$. Note that $G + H$ might not be a core but its core satisfies exactly the same inequalities in $(\vec{\mathscr{C}}, \leq)$. So the core of $G + H$ is the least upper bound of G and H. Analogously, it follows from Proposition 1.2.16 that the core of $G \times H$ is the greatest lower bound of G and H. □

Recall that the sum and product of two graphs is also a graph. Then, applying the previous proof to graphs, it follows that $(\mathscr{C}, \leq)$ is a sublattice of $(\vec{\mathscr{C}}, \leq)$.

Chapter 2

Density and Universality of the Homomorphism Order

2.1 Density and Gaps

Given a partial ordered set $(\mathcal{P}, \leq)$ and two elements $a, b \in \mathcal{P}$, let us write $a < b$ to mean $a \leq b$ and $b \nleq a$.

Let $a, b \in \mathcal{P}$ satisfying $a \leq b$. The *closed interval* $[a, b]$ is the set of elements $x \in \mathcal{P}$ such that $a \leq x \leq b$. Note that $[a, b]$ contains at least the elements a and b. The *open interval* (a, b) is the set of elements $x \in \mathcal{P}$ such that $a < x < b$. An open interval might be empty. For the purpose of this thesis we shall consider only closed intervals, and we shall refer to them just as *intervals*. An interval $[a, b]$ is a *gap* if there is no $x \in \mathcal{P}$ such that $a < x < b$, which is equivalent to say that the open interval (a, b) is empty.

A partially ordered set $(\mathcal{P}, \leq)$ is *dense* if for any pair of elements $a, b \in \mathcal{P}$ satisfying $a < b$ there exists an element $c \in \mathcal{P}$ such that $a < c < b$.

Proposition 2.1.1. *A partially ordered set is not dense if and only if it contains at least one gap.*

Proof. It is clear from the definitions of density and gap. $\square$

Observe that the partially ordered set $(\vec{\mathscr{C}}, \leq)$ is not dense since $[K_1, \vec{K_2}]$ is a gap. In other words, there is no digraph X satisfying $K_1 < X < \vec{K_2}$. This is true since $\vec{K_2} \nrightarrow X$ implies that $X \to K_1$, which is equivalent to say that a digraph X does not contain any arc if it is an independent set. However, the interesting question is not if the homomorphism order is dense but which classes of digraphs are. For instance, consider the class of graphs $(\mathscr{C}, \leq)$, which can also be seen as the class of symmetric digraphs. It is obvious that $(\mathscr{C}, \leq)$ is also not dense since there is no graph X satisfying $K_1 < X < K_2$, i.e. $[K_1, K_2]$ is a gap. But it was shown in [16] that this is the only gap, and otherwise $(\mathscr{C}, \leq)$ is dense.

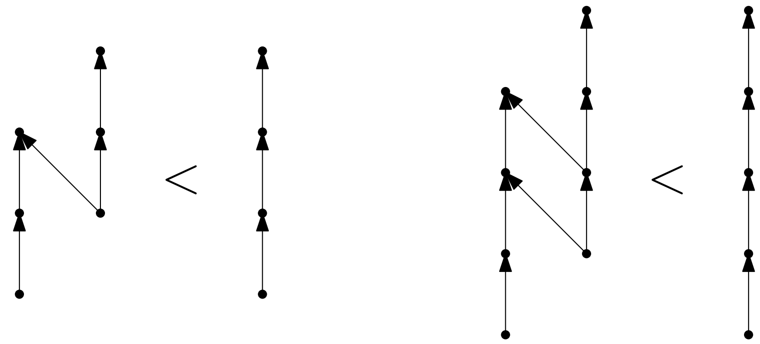

Figure 2.1: Example of two gaps in $(\vec{\mathscr{C}}, \leq)$. They correspond to the intervals $[\vec{T}_2 \times \vec{P}_3, \vec{P}_3]$ and $[\vec{T}_3 \times \vec{P}_4, \vec{P}_4]$ respectively.

The class of digraphs $(\vec{\mathscr{C}}, \leq)$ is not that simple as the class of graphs $(\mathscr{C}, \leq)$ as there are more than just one unique gap. But the good news is that those gaps were characterized by Nešetřil and Tardif [12]. It was shown that for every oriented tree T there exists a digraph G_T such that $[G_T, T]$ is a gap, and that all gaps have this form. An *oriented tree* is a connected acyclic digraph. We shall focus our interest on oriented trees in Chapter 3.

Theorem 2.1.2 (Characterization of the gaps [12])**.**

- *For every oriented tree T there exists a digraph G_T such that $[G_T, T]$ is a gap.*
- *All gaps in the homomorphism order of digraphs $(\vec{\mathscr{C}}, \leq)$ have the from $[G, T]$ where T is homomorphically equivalent to an oriented tree.*

It follows from Theorem 2.1.2 that every interval of the form $[G, H]$ where the core of H contains a cycle can not be a gap. Therefore there must exist a digraph X such that $G < X < H$. We now give a constructive proof of such digraph X. The following proof is a generalisation of [5, Theorem 3.32].

Theorem 2.1.3. *Let digraphs G, H be cores satisfying $G < H$, where H is connected and contains a cycle. Then there exists a digraph X such that $G < X < H$.*

Proof. Let ab be an arc belonging to some cycle in H. Let c be the other vertex in the cycle adjacent to a, which means that ac or ca is an arc of the cycle. Note that if the considered cycle in H is isomorphic to $\vec{C}_2$, then b and c would be the same vertex and both ab and ba would be arcs of H. Otherwise $b \neq c$ and ba is not an arc of H. Let H' be a digraph obtained from H by adding a new vertex a', and replacing the arc ab by the arc $a'b$. Note that if we identify the vertex a' with the vertex a we obtain the digraph H. See Figure 2.2.

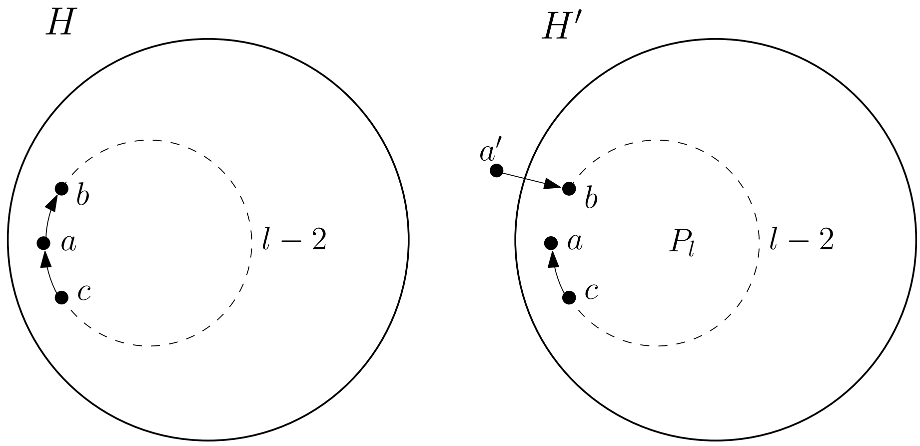

Figure 2.2: Digraph H and digraph H'. The arc joining the vertices a and c might be in the other direction.

Let $n > |G|$ and consider the complete graph K_n. Let X' be the digraph obtained from an arbitrary orientation of K_n by replacing each arc uv by a copy of H' identifying u with a' and v with a. Note that the vertices of K_n are in X'. We shall refer to them as *original vertices*.

Let $X = G + X'$. We claim that $G < X < H$.

It is clear that $G \to X$. Suppose now that $X' \to G$. Then, since $n > |G|$ and X' contains n original vertices, at least two original vertices from X' will be mapped to the same vertex in G. That will induce a homomorphism $H \to G$ since every pair of original vertices in X' are joined by a copy of H'. Thus, $X \nrightarrow G$.

Consider $f : X' \to H$ that maps each original vertex to a and the rest of vertices to their corresponding vertex in H. It is easy to check that f is a homomorphism. On the other hand, suppose there exists an homomorphism $g : H \to X'$. Since H is a core, $f \circ g : H \to H$ is an automorphism. Then there exists $h : H \to H$ such that $h = (f \circ g)^{-1}$. Consider now $g \circ h : H \to X'$. Since $f \circ g \circ h = id_H$, $g \circ h$ maps the vertex $a \in H$ to an original vertex $v_o \in X'$ and the vertex $b \in H$ to the vertex b of some copy $H'_o \subset X'$ such that $v_o \in H'_o$. Assume that $b \neq c$. It follows that the rest of vertices of the cycle will be mapped to their corresponding vertices of the same copy H'_o. Then $(g \circ h)(c)$ will not be adjacent to $v_o = (g \circ h)(a)$ but a and c are adjacent in H. This is a contradiction since $g \circ h$ is a homomorphism. In the case $b = c$, we shall have that $(g \circ h)(b)$ is not an inneighbour of $(g \circ h)(a)$ but ba is an arc in H, which is also a contradiction. So $H \nrightarrow X'$. Hence, $G < X < H$ as claimed. □

One might ask if this density result still holds in the class of connected cores. Although it is true that the digraph X obtained in Theorem 2.1.3 might not be connected, we shall present a technique which allows us to join its components and still have that $G < X < H$.

A *zig-zag* is a path which alternates forward and backward arcs. Observe that if a zig-zag has even length then the starting and ending vertex have the same level. On the other hand, if the length is odd the starting and ending vertex will have different level. The core of all zig-zags is the digraph $\vec{P}_1$ (or $\vec{K}_2$) consisting only on one arc.

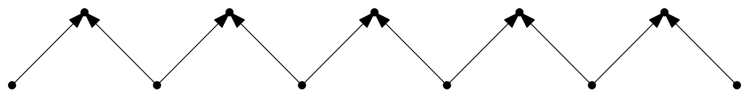

Figure 2.3: Zig-zag of length 10.

Given a digraph G and a zig-zag Z which starts in a vertex $v \in V(G)$, we say that the zig-zag is *proper* if there exists a homomorphism from Z to an arc of $G \backslash Z$.

Lemma 2.1.4. *Let digraphs G, X, H be cores satisfying $G < X < H$, where H is connected. Then there exists a connected digraph X', obtained from the joining of the components of X by proper and long enough zig-zags, such that $G < X' < H$.*

Proof. Assume that X is not connected, otherwise we are done. Consider a homomorphism $f : X \to H$. Let X_1, X_2 be two different components of X and let $x_1 \in X_1$ and $x_2 \in X_2$ be two vertices such that $d(f(x_1), f(x_2))$ is minimum. Consider the digraph obtained from X by adding two new vertices x_1', x_2', joining x_1 to x_1' and x_2 to x_2' by a proper zig-zag of even length greater than $|H|$, and joining x_1' to x_2' by the path P from $f(x_1)$ to $f(x_2)$ in H. Observe that f can be extended into a homomorphism from such digraph to H since the zig-zags have even length and can be mapped to an arc, implying that $f(x_1') = f(x_1)$ and $f(x_2') = f(x_2)$. Now, let X' be the connected digraph obtained by joining each pair of components in X by the previous procedure. Analogously, f can be extended into a homomorphism $f' : X' \to H$. It is clear that $G < X'$ since $X \subset X'$. Finally, suppose there exists a homomorphism $g : H \to X'$. Since every zig-zag has length greater than $|H$, then H must be homomorphic either to one of the components of X or to some path $P \subset H$. The first can not be since $H \nrightarrow X$, and the second is a contradiction since H is a core. Hence, $G < X' < H$. $\square$

2.2 Universal Intervals and the Fractal Property

Given two partially ordered sets $(\mathcal{P}_1, \leq_1)$ and $(\mathcal{P}_2, \leq_2)$, an *embedding* from $(\mathcal{P}_1, \leq_1)$ to $(\mathcal{P}_2, \leq_2)$ is a mapping $\Phi : \mathcal{P}_1 \to \mathcal{P}_2$ such that for every $a, b \in \mathcal{P}_1$, $a \leq b$ if and only if $\Phi(a) \leq \Phi(b)$. It follows from its definition that embeddings are injective mappings. If such a mapping exists we shall say that $(\mathcal{P}_1, \leq_1)$ can be embedded into $(\mathcal{P}_2, \leq_2)$.

A partially ordered set $(\mathcal{P}, \leq)$ is *universal* if every partially ordered set can be embedded into it. Recall that from the purpose of this thesis, we are only considering countable partially ordered sets.

The existence of universal partially ordered sets has been proved several times in the literature [3, 9, 4]. As a matter of fact, it has been shown that the class of graphs under the homomorphism order is universal [15]. Since $(\mathscr{C}, \leq)$ is a suborder of $(\vec{\mathscr{C}}, \leq)$, it follows that the class of digraphs under the homomorphism order is also universal.

Corollary 2.2.1. *$(\vec{\mathscr{C}}, \leq)$ is universal.*

Let $(\mathcal{P}, \leq)$ be a partially ordered set. We say that $(\mathcal{P}, \leq)$ has the *fractal property* if every interval $[a, b] \subseteq (\mathcal{P}, \leq)$ is either universal or a gap. Thus, if $[a, b]$ is a universal interval, every partially ordered set can be embedded into it. In particular, $(\mathcal{P}, \leq)$ can be embedded into $[a, b]$, which is the inspiration for this property to be called "fractal". This property was first introduced in [10]. Recently, it was shown that every interval in the homomorphism order of graphs $(\mathscr{C}, \leq)$ is universal, with the exception of $[K_1, K_2]$, which we know is the only gap in $(\mathscr{C}, \leq)$. Thus, it was shown that the homomorphism order of graphs has the fractal property.

One might ask if the same property holds for the class of digraphs. This case is clearly more complicated since there are infinitely many gaps, one for each oriented tree. We know by Theorem 2.1.3 that intervals of the form $[G, H]$ where the core of H contains a cycle are not gaps. In the remainder of this section, we shall prove that these intervals are indeed universal.

Observe that embedding a universal partially ordered set $(\mathcal{P}_1, \leq_1)$ into a partially ordered set $(\mathcal{P}_2, \leq_2)$, implies that $(\mathcal{P}_2, \leq_2)$ is also universal. For this reason, once you know the existence of many different universal partially ordered sets, see [8] for some examples, a simple way to show that a certain partially ordered set is universal is to construct an embedding from one of them into it.

Let $I(a, b)$ be a digraph I with two distinguished vertex a and b. For each digraph G let $\Phi_I(G)$ be the digraph obtained from G by replacing each arc $uv \in A(G)$ by a copy I_{uv} of I identifying the vertex u with a and the vertex v with b. Here, $I(a, b)$ is used as a gadget to construct a mapping from the class of all digraphs $\vec{\mathscr{C}}$ to some subclass of digraphs $\mathscr{S} \subseteq \vec{\mathscr{C}}$:

$$\Phi_I : \vec{\mathscr{C}} \to \mathscr{S} \subseteq \vec{\mathscr{C}}.$$

Observe that if we are able to show that Φ_I is a well-defined poset embedding then we are actually proving that the class $\mathscr{S}$ under the homomorphism order is universal. This is the technique we shall use for such proofs.

Lemma 2.2.2. *Let G_1, G_2 be two digraphs satisfying $G_1 < G_2$. Let $I(a,b)$ be a gadget. If for every digraph $F \in \vec{\mathscr{C}}$ the following conditions holds:*

(i) $G_1 < \Phi_I(F) < G_2$.

(ii) *Every homomorphism $f : I(a,b) \to \Phi_I(F)$ satisfies that $f(I) \subseteq I_{uv}$ for some arc $uv \in A(F)$, and $f(a) = a$ and $f(b) = b$.*

Then Φ_I is a poset embedding from $(\vec{\mathscr{C}}, \leq)$ into the interval $[G_1, G_2]$.

Proof. The condition *(i)* ensures that $\Phi_I : \vec{\mathscr{C}} \to [G_1, G_2]$ is a well-defined mapping.

Let F_1, F_2 be two digraphs and let $f : F_1 \to F_2$ be a homomorphism. Then f induces a homomorphism $g : \phi_I(F_1) \to \phi_I(F_2)$ by associating each arc in F_1 (or F_2) with its correspondent copy $I(a,b)$ in $\Phi_I(F_1)$ (or $\Phi_I(F_2)$), making the following diagram

$$\begin{array}{ccc} F_1 & \xrightarrow{f} & F_2 \\ \big\downarrow{\scriptstyle \Phi_I} & & \big\downarrow{\scriptstyle \Phi_I} \\ \Phi_I(F_1) & \xrightarrow{g} & \Phi_I(F_2). \end{array}$$

commutative.

Reciprocally, given a homomorphism $g : \Phi_I(F_1) \to \Phi_I(F_2)$, condition *(ii)* ensures that g sends each pair of vertices (a,b) in a copy $I(a,b) \subset \Phi_I(F_1)$ to the vertices (a,b) in a copy $I(a,b) \subset \Phi_I(F_2)$. As each copy $I(a,b)$ and hence also the pairs (a,b) are associated with arcs in F_1 or F_2, g induces a homomorphism $f : F_1 \to F_2$, and the above diagram is commutative.

Thus, $F_1 \to F_2$ if and only if $\Phi_I(F_1) \to \Phi_I(F_2)$, which shows that Φ_I is a poset embedding. ☐

Hence, in order to prove the universality of intervals of the form $[G, H]$ where the core of H contains a cycle, we are interested in finding an appropriate gadget $I(a,b)$ to ensure that $\Phi_I : \vec{\mathscr{C}} \to [G, H]$ satisfy the conditions of Lemma 2.2.2.

Lemma 2.2.3. *Let digraphs G, H be cores satisfying $G < H$ where H is connected and contains a cycle. Then there exists incomparable connected graphs X_1, X_2 such that $G < X_i < H$ for $i = 1, 2$.*

Proof. Consider the shortest cycle in H and let H' be the digraph from Figure 2.4. The construction of H' is analogous to the proof of Theorem 2.1.3.

Let $n > \max\{|G|, 3|H|\}$ and consider the complete graph K_n. Let X_1 be the digraph obtained from an arbitrary orientation of K_n by replacing each arc uv by a

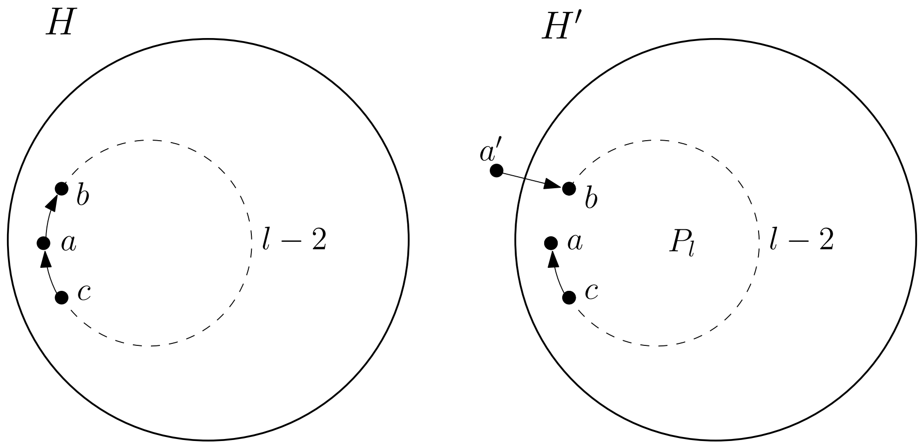

Figure 2.4: Digraph H, digraph H' and path P_l. The arc joining the vertices a and c might be in the opposite direction.

copy of H' identifying u with a' and v with a. Note that the vertices of K_n are in X_1. We shall refer to them as *original vertices*. It follows from the proof of Theorem 2.1.3 that $G < G + X_1 < H$.

Let S be a connected graph with chromatic number and girth greater than $\max\{|G|, |X_1|\}$. The existence of such graph is a well know result by Paul Erdős [1]. Let $\vec{S}$ be an orientation of S containing a directed cycle $\vec{C} \subset \vec{S}$. Let X_2 be a digraph obtained from $\vec{S}$ by replacing each arc uv by a copy of H' identifying u with a' and v with a. Analogously, it follows that $G < G + X_2 < H$.

Let us see that X_1 and X_2 are incomparable. Suppose $X_2 \to X_1$. Since $\chi(S) > |X_1|$, at least two original vertices of X_2 which are adjacent in S will be mapped to the same vertex in X_1. This induces a homomorphism $H \to X_1$, which is a contradiction. Suppose now that there exists a homomorphism $f : X_1 \to X_2$. Let $T \subset \vec{S}$ be the subgraph whose arcs are $\{xy \in A(\vec{S}) \mid \exists u \in X_1 \text{ such that } f(u) \in H'_{xy}\}$ where $H'_{xy} \subset X_2$ is the copy of H' corresponding to the arc xy. Since the girth of S is greater than $|X_1|$, T must be an oriented tree. Recall that by Proposition 1.2.8 for any balanced digraph T, in particular any tree, there exists a homomorphism $T \to \vec{P}_k$ for some $k > 0$, where $\vec{P}_k$ is the directed path of length k. Observe that $\vec{P}_k \to \vec{C} \subset \vec{S}$ for any $k > 0$. Then there exists a homomorphism $g : T \to \vec{C}$. Let T' and $\vec{C}'$ be the digraphs obtained by replacing each arc of T and $\vec{C}$ by a copy of H' respectively. Then g induces a homomorphism $g' : T' \to \vec{C}'$ identifying the arcs xy with their corresponding copies H'_{xy}. Observe that $\mathrm{Im}(f) \subset T'$ and $\vec{C}' \subset X_2$. Thus, $g' \circ f : X_1 \to X_2$ is a homomorphism. Let $v_o \in X_1$ be an original vertex and let $H'_o \subset \vec{C}'$ be a copy of H' such that $(g' \circ f)(v_o) \in H'_o$. Since any other original vertex $v_i \in X'_1$ is joined to v_o at least by a path of length l, $(g' \circ f)(v_i)$ must be mapped to H'_o or to one of the two copies of H' in $\vec{C}'$ that intersect H'_o. Hence,

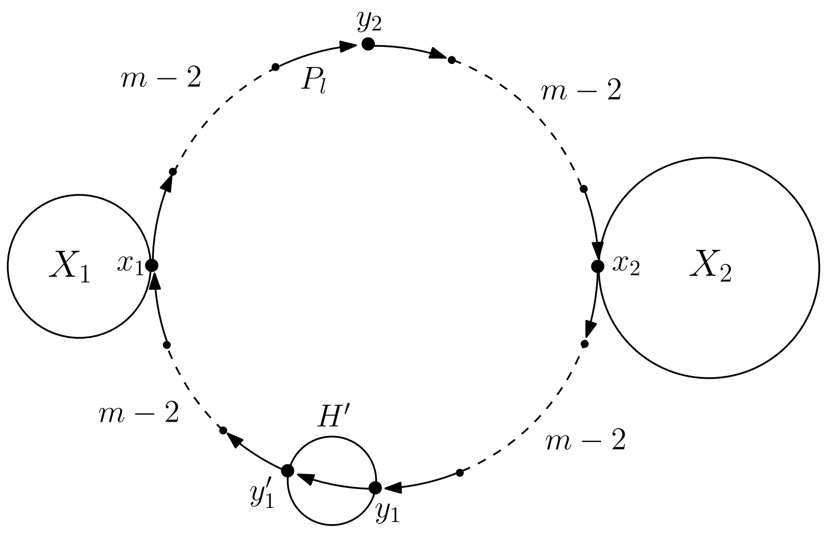

Figure 2.5: Digraph $I(y_1, y_2)$.

all original vertices of X_1 are mapped into at most three copies of H'. Considering that $\chi(K_n) > 3|H|$, it follows that at least two original vertices from X_1 will be mapped to the same vertex in X_2. This induces a homomorphism $H \to X_2$ which is a contradiction.

Finally, let X_1' be the joining of G with X_1 and let X_2' be the joining of G with X_2 as in Lemma 2.1.4. Then X_1' and X_2' are connected and incomparable digraphs such that $G < X_i' < H$ for $i = 1, 2$. □

Theorem 2.2.4. *Let digraphs G, H be cores satisfying $G < H$ where H is connected and contains a cycle. Then the interval $[G, H]$ is universal.*

Proof. Let X_1, X_2 be the digraphs of the proof of Lemma 2.2.3. Assume X_1, X_2 to be cores. Observe that X_1 and X_2 are connected since H is connected. We know that $G < G + X_i < H$ for $i = 1, 2$. Let $m > \max\{|X_1|, |X_2|\}$. Consider the path P_l from Figure 2.4 and let x_1 and x_2 be an original vertex of X_1 and $\vec{C}' \subseteq X_2$ respectively ($\vec{C}'$ is a digraph defined in the proof of Lemma 2.2.3). Let $I(y_1, y_2)$ be the digraph from Figure 2.5 which shall be used as the gadget. Observe that x_1 is joined to x_2 by two different paths, one consisting in $2m$ consecutive paths P_l and the other consisting in $2m + 1$ paths P_l in the opposite direction. Note that the vertices y_1, y_1' are in a copy of H' so y_1 is identified with a' and y_1' is identified with a. Observe that $I \backslash X_2 \to X_1$ and $I \backslash X_1 \to X_2$ due to the choice of x_1 and x_2. Finally, observe that $G < I < H$.

We claim that $I(y_1, y_2)$ satisfies the conditions of Lemma 2.2.2.

Given a digraph $F \in \vec{\mathscr{C}}$, let $\Phi_I(F)$ be the digraph obtained by replacing each arc $uv \in A(F)$ by a copy of I identifying u with y_1 and v with y_2. Observe that

$G < \Phi_I(F) < H$ and condition *(i)* holds.

Consider a homomorphism $f : I \to \Phi_I(F)$. Since X_1 and X_2 are connected incomparable cores, they must be mapped to a copy of itself in $\Phi_I(F)$ respectively. Suppose that the path P_l is not symmetric in respect to its middle point, so P_l has a direction. Then the only paths from the vertex x_1 of some copy X_1 to a vertex x_2 of some copy X_2 in $\Phi_I(F)$ consisting on $2m$ consecutive forward paths P_l are those from x_1 to x_2 of the same copy of I. Thus, $f(x_1) = x_1$, $f(x_2) = x_2$ and $f(y_2) = y_2$. And the same happens with the path from x_1 to x_2 consisting on $2m+1$ consecutive backward paths P_l, so $f(y_1) = y_1$. On the other hand, suppose that the path P_l is symmetric in respect its middle point. If there exists a homomorphism $H' \to P_l$, then the core of H, and hence H, must be a cycle (the one obtained by identifying the starting and the ending vertices of P_l). However, since P_l is symmetric, this implies that H can be collapsed into a path, which is a contradiction. So $H' \nrightarrow P_l$. Then the only pair of vertices, one of some copy of X_1 and the other of some copy of X_2 in $\Phi_I(F)$, that are joined at the same time by a path consisting on $2m$ consecutive paths P_l, and by a path consisting on $2m+1$ consecutive paths P_l but containing a copy of H', are the vertices x_1 and x_2 of the same copy of I. Thus, $f(x_1) = x_1$, $f(x_2) = x_2$, $f(y_2) = y_2$ and $f(y_1) = y_1$. We conclude that any homomorphism $f : I \to \Phi(F)$ maps $I(y_1, y_2)$ to some copy of it in $\Phi(F)$ fixing the vertices y_1 and y_2. Hence, condition *(ii)* holds.

It follows by Lemma 2.2.2 that Φ_I is an embedding from $(\vec{\mathscr{C}}, \leq)$ into the interval $[G, H]$, and thus, $[G, H]$ is universal. $\square$

We have proved that every interval in $(\vec{\mathscr{C}}, \leq)$ of the form $[G, H]$ where the core of H contains a cycle is universal. The remaining cases are the intervals $[G, T]$ where the core of T is an oriented tree. This cases are more complicated since there is no density theorem for them. In fact, every gap $[G, T]$ in the homomorphism order of digraphs satisfies that T is an oriented tree. In the next chapter, we are focusing our interest on the class of oriented trees.

Chapter 3

The Class of Oriented Trees

3.1 Oriented Paths and Trees

As we have already seen in the previous chapters, all gaps in the homomorphism order have the form $[G, T]$ where the core of T is a tree. We proved that all remaining intervals, which are those of the form $[G, H]$ where the core of H contains a cycle, are universal. In this chapter we will study not the intervals $[G, T]$ in the homomorphism order of digraphs but intervals $[T_1, T_2]$ in the homomorphism order of trees. Perhaps surprisingly, we shall show that if the trees have height greater or equal to 4 then every interval $[T_1, T_2]$ is universal. Moreover, we shall prove that even the class of oriented paths satisfies such property. Trees of height smaller than 4 have a simple structure and they are easily characterized (see Propositions 3.1.6 and 3.1.8 below). In this section we will present some of the concepts and results we will need to prove the main theorems of Chapter 3. Let us start by recalling some basic definitions.

An *oriented path* is a digraph consisting in a sequence of different vertices $P = (v_0, \dots, v_n)$ together with a sequence of different arcs $A(P) = (a_1, \dots, a_n)$ such that a_i is either the arc $v_{i-1}v_i$ or the arc v_iv_{i-1} for each $i = 1, \dots, n$. We say that n is the *length* of the path. An *oriented cycle* its defined analogously to a path but with $v_0 = v_n$. An *oriented tree* is a connected digraph containing no cycles. In what follows we only consider oriented paths, oriented cycles and oriented trees and we refer to them simply as paths, cycles and trees. Note that, in particular, a path is a tree, so every definition and property which applies to trees is also valid for paths.

Proposition 3.1.1. *A digraph is a tree if and only if every pair of vertices is joined by a unique path.*

Proof. Let G be a digraph. Suppose G contains a cycle C. Let $v_0, v_1, \dots, v_n = v_0$ be the sequence of vertices of C, then v_0, v_1 and $v_1, \dots, v_n = v_0$ are two different paths between v_0 and v_1. Suppose G has a pair of different vertices a, b which are joined by two different paths $a = v_0, \ldots, v_n = b$ and $a = u_0, \ldots, u_m = b$. We have $v_0 = u_0$ so consider the minimum $i > 0$ such that $v_i \neq u_i$, which exists since the paths are different. Consider now the minimum $j > i$ such that $v_j = u_j$, which exists again since $v_n = u_m$. Then $v_{i-1}, \ldots, v_j = u_j, \ldots, u_{i-1} = v_{i-1}$ is a cycle in G. □

The *height* of a tree is the maximum difference between forward and backward arcs of a subpath in it. Recall from Proposition 1.2.8 that, since every tree T is a balanced digraph, there exists a homomorphism $f : T \to \vec{P}_k$ for some integer $k > 0$. So given a tree T, consider the minimum $k > 0$ for which f exists. Consider $\vec{P}_k$ as the path with vertices $0, 1, \ldots, k$ and arcs $01, 12, \ldots, (k-1)k$. The *level* of a vertex $v \in T$ is the integer $f(v)$. Thus, the height of T can also be defined as the number k. Recall that, by Corollary 1.2.9, every homomorphism between trees preserves the level of vertices. This implies the following proposition.

Proposition 3.1.2. *Let T_1 and T_2 be two trees. If $f : T_1 \to T_2$ is a homomorphism then the height of T_1 is less or equal to the height of T_2.*

We define the level of an arc as the lowest level of its incident vertex. So the arc vu has level $l(vu) = l(v) = l(u) - 1$. Recall that if T is a tree of height k then for any $v \in T$, $0 \leq l(v) \leq k$. It follows that every arc vu in T has level $0 \leq l(vu) \leq k - 1$. Thus, Corollary 1.2.9 implies the following.

Proposition 3.1.3. *If T_1 and T_2 are two trees of the same height, then any homomorphism from T_1 to T_2 preserves the levels of vertices and arcs.*

A *leaf* is a vertex of a tree of degree one.

Proposition 3.1.4. *Let T be the core of a tree and let $v \in T$ be a leaf. If vu is an arc of T then v is the only inneighbour of u. If uv is an arc of T then v is the only outneighbour of u.*

Proof. Suppose vu and wu are two different arcs of T. Then the mapping $f : T \to T$ defined as $f(v) = w$ and $f(x) = x$ for the rest of vertices is a homomorphism which is not injective. The other case is analogous. □

Observe that in particular, Proposition 3.1.4 implies that the core of a path starts and ends with two arcs in the same direction, with the only exception of the path $\vec{P}_1$ (or $\vec{K}_2$) which is the digraph consisting of one arc.

Proposition 3.1.5. *The core of a path starts and ends with two arcs in the same direction.*

In order to study the homomorphism order of the class of trees, we shall be interested in how the core of a tree looks like and which properties may it have. We start by characterizing all balanced digraphs of height smaller or equal to 3.

Proposition 3.1.6. *$\vec{P}_1$ and $\vec{P}_2$ are the only cores of a balanced digraph of height one and two respectively. As a result, $[\vec{P}_1, \vec{P}_2]$ is a gap in $(\vec{\mathscr{C}}, \leq)$.*

In particular, if we denote by $\mathscr{P}$ the class of oriented paths and by $\mathscr{T}$ the class of oriented trees, $[\vec{P}_1, \vec{P}_2]$ is also a gap in $(\mathscr{P}, \leq)$ and $(\mathscr{T}, \leq)$.

Let L_k be the path of height three given in Figure 3.1. We consider a and d to be the initial and ending vertex of L_k respectively. Note that L_k is a core for every $k \geq 0$ and $L_0 = \vec{P}_3$.

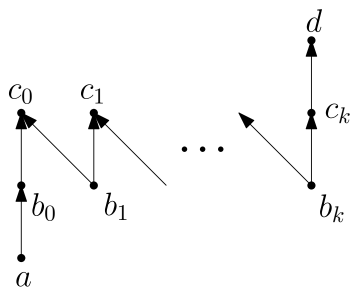

Figure 3.1: The path L_k.

Proposition 3.1.7. *$L_k \leq L_l$ if and only if $k \geq l$.*

Proof. It follows by the fact that homomorphisms preserve adjacency and the level of vertices. □

Proposition 3.1.8. *The core of a balanced digraph of height equal to three is equal to L_k for some $k \geq 0$.*

Proof. Consider a digraph G of height equal to three and let P be a path of minimum length joining a vertex of level 0 to a vertex of level 3. It can be seen that $P = L_k$ for some $k \geq 0$. See figure 3.1. We define a homomorphism $f : G \to P$ as follows. For every vertex $v_0 \in G$ of level 0, $f(v_0) = a$. For every vertex $v_3 \in G$ of level 3, $f(v_3) = d$. For every vertex $v_1 \in G$ of level 1, let v_0 be the closest vertex to v_1 of level 0. Then $f(v_1) = b_i$ where i is the maximum $0 \leq i \leq k$ such that $d(b_i, a) \leq d(v_1, v_0)$. Finally, for every vertex $v_2 \in G$ of level 2, let v_0 be the closest vertex to v_2 of level 0. Then $f(v_2) = c_i$ where i is the maximum $0 \leq i \leq k$ such that $d(c_i, a) \leq d(v_2, v_0)$. It follows that L_k is the core of G. □

The previous propositions, together with the fact that every digraph homomorphic to a balanced digraph is also balanced, implies the following result:

Proposition 3.1.9. *$[L_{k+1}, L_k]$ is a gap of $(\vec{\mathscr{C}}, \leq)$, for any $k \geq 0$.*

Let us denote $\mathcal{L}$ to the set of L_k for every $k \geq 0$. Observe that $(\mathcal{L}, \leq)$ is a linear order. In fact, $(\mathcal{L}, \leq)$ is isomorphic to the natural order of negative integers by associating each negative integer $(-k)$ with the path L_{k-1}. Observe that the existence of $(\mathcal{L}, \leq)$ already excludes the homomorphism order of digraphs of having the fractal property as we have defined in section 2.2. Intervals as $[L_2, \vec{P}_3]$, or $[\vec{P}_2, \vec{P}_3]$, are neither a gap nor universal.

As we have seen, the homomorphism order of trees of height less or equal to three is really simple. First there is the gap $[\vec{P}_1, \vec{P}_2]$, and then there is the linear order $(\mathcal{L}, \leq)$ in the interval $[\vec{P}_2, \vec{P}_3]$. Although the homomorphism order of trees of height greater or equal to four is more complex, we will find a really useful property in terms of homomorphisms that all trees share. To do so, let us define a new term related to trees. Given a tree T, a vertex $u \in T$ and a set of vertices $S \subseteq T$, the *plank* from u to S, denoted $P(u, S)$, is the subgraph induced by the vertices of every path which starts with u and contains some vertex $v \in S$.

Figure 3.2: A tree T at the left and the plank $P(u, \{x\})$ at the right.

Lemma 3.1.10. *Let T be a tree and let $v, u \in T$ be adjacent vertices. If $f : T \to T$ is an automorphism then $P(u, \{v\})$ is isomorphic to $P(f(u), \{f(v)\})$.*

Proof. Recall that if f is an automorphism then there exists an homomorphism $f^{-1} : T \to T$ such that $f \circ f^{-1}$ is the identity mapping.

First let us see that $f(P(u, \{v\}) \subseteq P(f(u), \{f(v)\})$. Suppose there exists a vertex $x \in P(u, \{v\})$ such that $f(x) \notin P(f(u), \{f(v)\})$. Then the path joining $f(x)$ to $f(u)$ does not contain the vertex $f(v)$. But applying f^{-1} to such path will imply that the path joining x to u neither contains the vertex v. This is a contradiction since $x \in P(u, \{v\})$.

Finally let us show that the number of vertices of $P(u, \{v\})$ is equal to the number of vertices of $P(f(u), \{f(v)\})$. This would imply that $f|_{P(u,\{v\})} : P(u, \{v\}) \to P(f(u), \{f(v)\})$ is in fact an isomorphism.

Suppose there exists a vertex $x \in V(T)$ such that $f(x) \in P(f(u)\{f(v)\})$ but $x \notin P(u, \{v\})$. Then the path joining $f(x)$ to $f(u)$ contains the vertex $f(v)$. But, as before, applying f^{-1} to such path will imply that the path joining x to u contains the vertex v, which is a contradiction since $x \notin P(u, \{v\})$. $\square$

Recall that a digraph G is *rigid* if it is a core and the only automorphism $f : G \to G$ is the identity mapping.

Lemma 3.1.11. *The core of a tree is rigid.*

Proof. Let T be the core of a tree. Let $f : T \to T$ be a homomorphism. Recall that f must be an automorphism since T is a core. Suppose f is different from the identity on T and let $u = \min_{v \in T}\{d(v, f(v)) | v \neq f(v)\}$.

Let $u = v_0, v_1, \ldots, v_k = f(u)$ be the path that joins u with $f(u)$. Observe that $k \neq 1$, since otherwise u and $f(u)$ would be adjacent implying that u and $f(u)$ have different levels, but f is level preserving. Note also that $f(v_1)$ is adjacent to $f(u)$.

First we want to show that $f(v_1) = v_{k-1}$. Let us suppose that $f(v_1) \neq v_{k-1}$. Since f is an automorphism, $P(u, \{v_1\})$ is isomorphic to $P(f(u), \{f(v_1)\})$ by Lemma 3.1.10, and therefore $|P(u, \{v_1\})| = |P(f(u), \{f(v_1)\})|$. But $P(f(u), \{f(v_1)\}) \subset P(u, \{v_1\})$, which is a contradiction. It follows that $f(v_1) = v_{k-1}$. Now, let us consider two cases.

Suppose $k > 2$. Then $v_1 \neq v_{k-1}$. Observe that v_1 satisfies that $v_1 \neq f(v_1)$ and $d(v_1, f(v_1)) = k - 2$. But this is a contradiction, since k was the minimum such distance.

Suppose $k = 2$. Then $v_1 = v_{k-1} = f(v_1)$. By Lemma 3.1.10, $P(v_1, \{u\})$ is isomorphic to $P(v_1, \{f(u)\})$. Let $g : T \to T$ be a mapping equal to the identity on $T \backslash P(v_1, \{u\})$ and equal to f on $P(v_1, \{u)$. It is clear that $g : T \to T$ is a homomorphism since $f(v_1) = v_1$. So g must be an automorphism. However, we have reached a contradiction because then g is not injective since $g(u) = g(f(u))$. $\square$

This is a strong result since it allows us to apply the properties of Proposition 1.3.7 to the core of every tree. For this reason, Lemma 3.1.11 will be essential for every proof in the next sections.

3.2 Density for Paths

In order to prove a density theorem and characterize the universal intervals of the homomorphism order of the class of oriented trees we shall first look to a simpler case which is the class of paths. Oriented paths is probably one of the simplest cases of digraphs. Homomorphisms between oriented paths have been studied in [6, 13, 5] and surprising results have been found. For instance, Nešetřil and Zhu showed in [13] a density theorem for paths of height greater or equal to four. In this section, we will prove a slightly more general version of this theorem and we will characterize all universal intervals in $(\mathscr{P}, \leq)$. Theorem 3.2.1 is one of the main contributions of this work.

For a path $P = (v_1, \ldots, v_n)$ we denote by $i(P) = p_0$ and $t(P) = p_n$ the initial and terminal vertex of P respectively. We write $P^{-1} = (v_n, v_{n-1}, \ldots, v_0)$ for the reverse path of P, where vu is an arc of P^{-1} if and only if uv is an arc of P. That is, P^{-1} is the path obtained from P by changing the direction of all its arcs. For two paths $P = (v_0, v_1, \ldots, v_n)$ and $P' = (v'_0, v'_1, \ldots, v'_{n'})$, the concatenation PP' is the path $PP' = (v_0, \ldots, v_n = v'_0, v'_1, \ldots, v'_{n'})$ with the induced arcs by P and P'.

Theorem 3.2.1. *[Density] Let T_1 be a tree and P_2 a path such that $T_1 < P_2$. If the height of P_2 is greater or equal to 4, then there exists a tree T satisfying $T_1 < T < P_2$.*

Proof. Assume that T_1 and P_2 are cores. Note that T_1 is either a path or a proper tree. Let

$$f : T_1 \to P_2$$

be a homomorphism. We also assume that $f(T_1)$ is a core. Thus, $f(T_1)$ starts and ends with two arcs in the same direction. We now consider two cases.

Case 1: The mapping f is not surjective.

In this case $f(T_1) \subsetneq P_2$, so either the initial vertex or the terminal vertex of P_2 does not belong to $f(T_1)$. Suppose without loss of generality that $t(P_2) \notin f(T_1)$. Let A and B be the paths from $i(P_2)$ to $t(f(T_1))$ and from $t(f(T_1))$ to $t(P_2)$ respectively. Note that A and B are non empty paths, $f(T_1) \subseteq A$ and $P_2 = AB$. Let B^* be the subpath of P_2 consisting of B and the last arc of A. Consider the concatenation path AZB where Z is a zig-zag of level equal to the last arc of A and even length greater than $\max\{|T_1|, |P_2|\}$. We have that $AZ \to A$ and $ZB \to B^*$.

We now show that $T_1 < AZB < P_2$.

It is clear that $T_1 \to AZB \to P_2$. Suppose that $P_2 \to AZB$. Since $|Z| > |P_2|$, either $P_2 \to AZ \to A$ or $P_2 \to ZB \to B^*$, and both cases contradicts the fact that P_2 is a core since both A and $B*$ are proper subpaths of P_2. Thus, $P_2 \nrightarrow AZB$.

To show that $AZB \nrightarrow T_1$, suppose that there exists a homomorphism $g : AZB \to T_1$. We then have $T_1 \to f(T_1) \to T_1$ which must be equal to the identity mapping since T_1 is a core. Thus, $f(T_1)$ is isomorphic to T_1 so we can consider g as a homomorphism $g : AZB \to f(T_1)$. Note that the restriction $g|_{f(T_1)}$ must also be the identity mapping. Since $f(T_1)$ ends with two arcs in the same direction, it follows that g must collapse the zig-zag Z to the last arc of $f(T_1)$. Thus, g induces a homomorphism $AB \to f(T_1)$ contradicting that $AB = P_2 \nrightarrow T_1$.

Case 2: The mapping f is surjective.

Observe that the mapping f cannot be one to one. Indeed, if T_1 is a proper tree then $f(T_1) = P_2$ would be also a proper tree, else if T_1 is a path then f^{-1} would be a homomorphism from P_2 to T_1, contrary to our assumption that $P_2 \nrightarrow T_1$.

Thus, there must exists two different vertex $v_1, v_2 \in T_1$ such that $f(u_1) = f(u_2)$. Actually, it is not difficult to see that there exists a pair of vertices $v_1, v_2 \in T_1$ with a common neighbour v_0 such that $f(v_1) = f(v_2)$, so let us assume that this is the case. Then we have $l(v_1) = l(v_2)$, and either both v_1v_0 and v_2v_0 are arcs of T_1 or both v_0v_1 and v_0v_2 are arcs of T_1. By inverting all orientations if necessary we may assume that v_1v_0 and v_2v_0 are arcs of T_1.

Let T' be the tree obtained from T_1 by identifying the vertices v_1 and v_2. We denote the new vertex, which is the identification of v_1 and v_2, by v. It is straightforward to see that there exists a homomorphism

$$f' : T' \to P_2$$

induced by f, such that $f'(v) = f(v_1) = f(v_2)$ and $f' = f$ for the rest of vertices.

It follows that $T_1 \to T' \to P_2$. Note that it can not happen that $T' \to T_1$ since then there would be an automorphism $T_1 \to T' \to T_1$ equal to the identity mapping because T_1 is a core, but $|T'| < |T_1|$. Hence, if $P_2 \nrightarrow T'$ we have $T_1 < T' < P_2$ which concludes the proof.

So for the remaining part of the proof, we shall assume that there exists a homomorphism

$$g : P_2 \to T'$$

.

Observe that the homomorphism $(f' \circ g) : P_2 \to T' \to P_2$ is an automorphism and therefore it must be equal to the identity mapping since P_2 is a core. It follows that g is injective and $g(P_2)$ is isomorphic to P_2. Note that since P_2 is a core and $T' \to P_2$, $g(P_2)$ is the core of T'.

Let $V_1 = P(v_0, v_1)\backslash\{v_0\}$ and $V_2 = P(v_0, v_2)\backslash\{v_0\}$ be subtrees of T_1 and T'. See Figure 3.3. To see that V_1 and V_2 are non empty, observe that if V_1 were empty it would exists a homomorphism $T_1 \to T_1$, that maps the arc v_1v_0 to the arc v_2v_0, which is different from the identity mapping contradicting that T_1 is a core. And the same argument applies to V_2.

We have seen that $g(P_2)$, which is isomorphic to P_2, is the core of T'. Then $g(P_2)$ must intersect both V_1 and V_2 in some vertices besides v. For otherwise $g(P_2)$ would be also contained in T_1 but $P_2 \nrightarrow T_1$. Since $g(P_2)$ is a path intersecting V_1 and V_2, it must contain the vertex v. Let $v_1' \in V_1$ and $v_2' \in V_2$ be the adjacent vertices to v in $g(P_2)$. The restriction $f'|_{g(P_2)} : g(P_2) \to P_2$ identifies the vertices of $g(P_2)$ with the vertices of P_2 as the identity mapping. Then we have $f(v_1') = f'(v_1') = v_1'$,

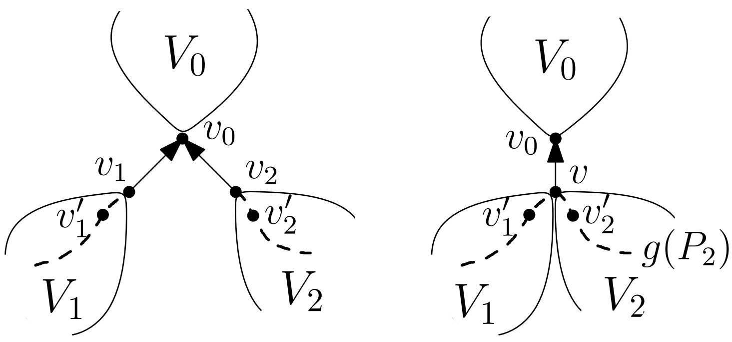

Figure 3.3: Tree T_1 and tree T'. The dashed path in T' represents $g(P_2)$ which is isomorphic to P_2.

$f(v_1) = f(v_2) = f'(v) = v$ and $f(v_2') = f'(v_2') = v_2'$, where the images v_1', v, v_2' are seen as vertices of P_2. See Figure 3.3. Finally, observe that $f(v_0)$ is either equal to v_1' or v_2'. Without loss of generality we shall assume that $f(v_0) = v_2'$, so v_2v_2' is an arc of P_2 and T_1.

In order to finish the proof, we shall now consider two different cases according to the direction of the arc incident to v_1 and v_1'.

Case 2.a: v_1v_1' is an arc of T_1.

Let w be the non terminal vertex of P_2, of level different to $l(v_1)$ and $l(v_2)$, which is closest to v_1' or v_2'. We know such vertex exists since the height of P_2 is greater or equal to 4. Without loss of generality, assume that w is at the right of v_2' in the sequence of the path P_2.

Let A the path from $i(P_2)$ to v_1, let B be the path from v_1 to v_2, let C be the path from v_2 to w and let D be the path from w to $t(P_2)$. Note that D is non empty since $w \neq t(P_2)$. We have that $P_2 = ACD$, and $ABCD$ is a subpath in T_1. Finally, let D^* be the subpath of P_2 consisting of D and the last arc of C.

We construct now a tree T obtained from two copies of T_1, which we denote by T_1^1 and T_1^2, joined as in Figure 3.4: we join the vertex $t(g(P_2))$ from the copy T_1^1 to the vertex v_1 from the copy T_1^2 by the path $D^{-1}ZC^{-1}$, where Z is a zig-zag of level equal to the last arc of C and even length greater than $|P_2|$. Let us write $T_1^1D^{-1}$ to express the joining of the tree T_1^1 with the path D^{-1} by the vertex $t(g(P_2))$. Analogously, let us write $C^{-1}T_1^2$ to express the joining of C^{-1} and T_1^2 by the vertex v_1. According to this notation we have that $T = T_1^1D^{-1}ZC^{-1}T_1^2$.

We claim that $T_1 < T < P_2$.

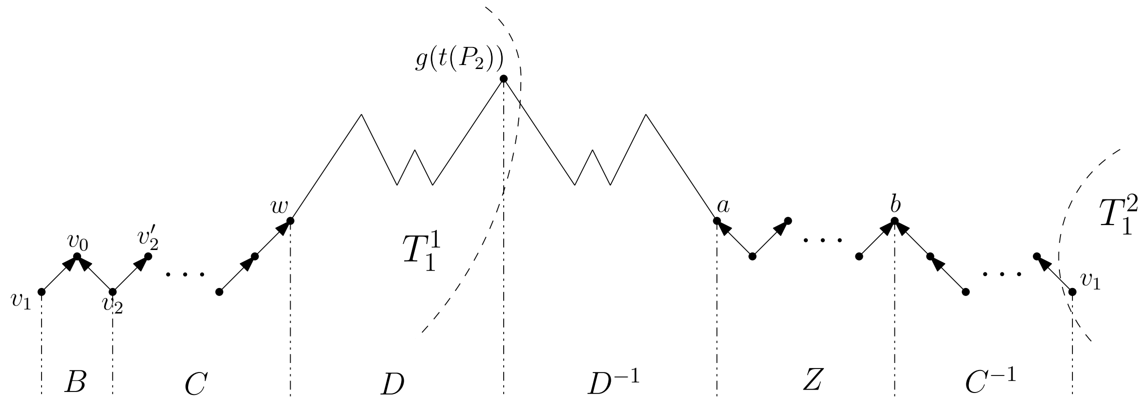

Figure 3.4: Example of how tree T from Case 2.a might look. The dashed lines indicate the border of the trees T_1^1 and T_1^2. The subpath $(D^*)^{-1}$ consists on the path D^{-1} and the first arc of Z.

Observe that there is a homomorphism

$$h : T \to P_2$$

such that $h|_{T_1^i} = f : T_1 \to P_2$ for $i = 1, 2$, the vertices of C^{-1} and D^{-1} are mapped to its correspondent vertices in $C, D \subset P_2$, and the zig-zag Z is collapsed into the last arc of $C \subset P_2$. Note that the homomorphism h is well defined since $f(v_2) = f(v_1)$.

Then, it is clear that $T_1 \hookrightarrow T \to P_2$.

To show that $P_2 \nrightarrow T$, we assume to the contrary that there is a homomorphism $\rho : P_2 \to T$. It follows by the fact that $|Z| > |P_2|$ that either *(a)* $P_2 \to T_1^1(D^*)^{-1}$ or *(b)* $P_2 \to C^{-1}T_1^2$. The case *(a)* can not happen since $T_1^1(D^*)^{-1} \to T_1^1$ but $P_2 \nrightarrow T_1$. Suppose now that *(b)* happens, so $\rho : P_2 \to C^{-1}T_1^2$ is a homomorphism. If $\rho(P_2) \cap C^{-1} = \emptyset$, then $\rho(P_2) \subset T_1^2$ but $P_2 \nrightarrow T_1$. Thus, it must happen that $\rho(P_2)$ and C^{-1} intersect. Recall that $T \to P_2$, so in particular $C^{-1}T_1^2 \to P_2$. It follows that ρ must be injective and $\rho(P_2)$ is isomorphic to P_2. Then either $\rho(i(P_2))$ or $\rho(t(P_2))$ is in C^{-1}. However, if this happens, the composition $(h \circ \rho) : P_2 \to P_2$ would not be equal to the identity since $i(P_2), t(P_2) \notin C$ in P_2. Hence, we must have $P_2 \nrightarrow T$.

We now conclude the Case 2.a by showing that $T \nrightarrow T_1$. Suppose that $\pi : T \to T_1$ is a homomorphism. Since T_1 is a core we have that T_1 is the core of T and $\pi|_{T_1^i} : T_1^i \to T_1$ is the identity mapping for $i = 1, 2$. Let $a, b \in T$ be the vertices $a = D^{-1} \cap Z$ and $b = Z \cap C^{-1}$. See Figure 3.4. We now consider π as a homomorphism $\pi : T \to T_1^1$ and we look how π could map the vertices of T to the vertices of T_1^1. First, we know that $\pi(T_1^1) = T_1^1$ via the identity mapping. We look now to $\pi(a)$. Such vertex must be at distance less or equal to $|D^{-1}|$ of the vertex $t(g(P_2))$. So the closest vertex to $v_1 \in T_1^1$ in which a can be mapped is $\pi(a) = w$. It follows that the closest arc to v_1 in which Z can be collapsed is the last arc of C.

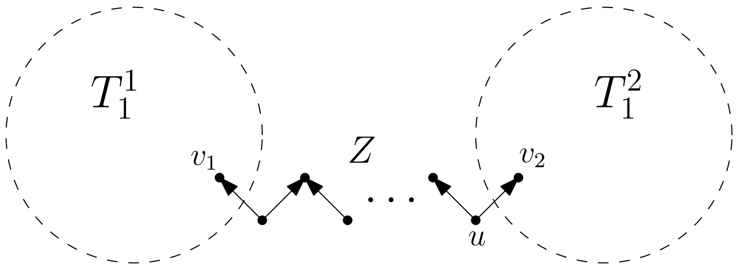

Figure 3.5: Tree T from case 2.b.

Then, the closest vertex to v_1 in which b can be mapped is $\pi(b) = w$. Consider now the vertex $v_1 \in T_1^2$, which is at distance $|C^{-1}|$ from b. Since $\pi|_{T_1^2}$ is as the identity mapping, we must have that $\pi(v_1) = v_1$. However $\pi(b) = w$ is at least at distance $|BC|$ from $v_1 \in T_1^1$, which is a contradiction. Thus, $T \nrightarrow T_1$.

Case 2.b: $v_1'v_1$ *is an arc of* T_1.

First we show that we can assume that v_1' is not the initial vertex of P_2. (In the dual case, i.e. in the case that v_1v_1', v_1v_0, v_2v_0 and $v_2'v_2$ are arcs of T_1 we would need to assume that v_2' is not the terminal vertex of P_2.) Suppose that $v_1' = i(P_2)$. We choose the vertices v_0, v_2, v_2' to play the role of the vertices v_1, v_0, v_2 respectively. Let v_2'' be the adjacent vertex to v_2', other than v_2, in the sequence of P_2. If $v_2''v_2'$ is an arc of P_2 we are in a case which is symmetric, up to orientation, to Case 2.a so the arguments in that case apply. Else, if $v_2'v_2''$ is an arc of P_2 then the case is symmetric to Case 2.b, with the difference that this time v_2'' can not be the terminal vertex of P_2 since the path from $i(P_2) = v_1'$ to v_2'' would have height 3 but P_2 have height greater or equal to 4. Thus, we can assume without loss of generality that v_1' is not the initial vertex of P_2.

Let T be the tree obtained from two copies of T_1, which we denote by T_1^1 and T_1^2, joined as in Figure 3.5: we join the vertex v_1 from the copy T_1^1 to the vertex v_2 from the copy T_1^2 by a zig-zag Z of level equal to the level of $v_1'v_1$ and even length greater than $|P_2|$. We use the same notation as in the previous case and write $T = T_1^1 Z T_2^2$.

We claim that $T_1 < T < P_2$.

Observe that there is a homomorphism

$$h : T \to P_2$$

such that $h|_{T_1^i} = f : T_1 \to P_2$ for $i = 1, 2$, and the zig-zag Z is collapsed into the arc $v_1'v_1$ in P_2. Note that h is well defined since $f(v_1) = f(v_2)$.

It is clear that $T_1 \hookrightarrow T \to P_2$.

Now we show that $P_2 \nrightarrow T$. Otherwise suppose that $P_2 \to T$. Then, since

$|Z| > |P_2|$, we have either *(a)* $P_2 \to T_1^1 Z$ or *(b)* $P_2 \to ZT_1^2$. Case *(a)* can not happen since $T_1^1 Z \to T_1^1$ but $P_2 \nrightarrow T_1$. Suppose that *(b)* holds, so there is a homomorphism $\rho : P_2 \to ZT_1^2 \to T_1*$, where T_1* is the tree consisting of T_1^2 and the last arc of Z, which we denote as uv_2. See Figure 3.5. Since $P_2 \nrightarrow T_1$ we must have that $u \in \rho(P_2)$. By considering the composition $(h \circ \rho) : P_2 \to P_2$ we see that ρ must be injective and $\rho(P_2)$ is isomorphic to P_2. Thus, u is either the initial or the terminal vertex of $\rho(P_2)$. But $h(u) = v_1' \neq i(P_2), t(P_2)$, so the automorphism $(h \circ \rho)$ is not equal to the identity mapping contradicting that P_2 is a core.

Finally, we end the proof by showing that $T \nrightarrow T_1$. On the contrary, suppose that $\pi : T \to T_1$ is a homomorphism. Then $\pi|_{T_1^i} : T_1^i \to T_1$ is the identity mapping for $i = 1, 2$, since T_1 is a core. Consider the vertices $v_1 \in T_1^1$ and $v_2 \in T_1^2$. We have that $\pi(v_1) = v_1$ and $\pi(v_2) = v_2$. It follows that $\pi(Z)$ must be a zig-zag of level $l(v_1'v_1)$ joining $\pi(v_1)$ and $\pi(v_2)$. However, there is no such zig-zag joining v_1 and v_2 in T_1. Hence, $T \nrightarrow T_1$. □

Note that given an interval $[P_1, P_2]$ of two paths, Theorem 3.2.1 ensures the existence of a tree T such that $P_1 < T < P_2$. However, T might not be a path. To obtain a density theorem for paths we shall need a result showed by Hell and Zhu.

Theorem 3.2.2 ([6])**.** *Let G be a digraph and P and oriented path. Then $G \nrightarrow P$ if and only if there is an oriented path P' such that $P' \to G$ and $P' \nrightarrow P$.*

The following theorem is originally proved in [13]. Here, we prove it as a corollary of Theorem 3.2.1.

Theorem 3.2.3. *Let P_1 and P_2 be two paths such that $P_1 < P_2$. If the height of P_2 is greater or equal to 4, then there exists a path P satisfying $P_1 < P < P_2$.*

Proof. By theorem 3.2.1 there exists a tree T such that $P_1 < T < P_2$. Since $T \nrightarrow P_1$, by theorem 3.2.2 there exists a path P' such that $P' \to T$ and $P' \nrightarrow P_1$. Let $f : P_1 \to T$ and $f' : P' \to T$ be homomorphisms. Let P be the path obtained from P_1 and P' joining the vertex $t(P_1)$ with the vertex $i(P')$ by the path from $f(t(P_1))$ to $f'(i(P'))$ in T. By construction $P_1 < P \leq T$. Hence, $P_1 < P < P_2$. □

3.3 Paths and the Fractal Property

We have characterized all gaps in $(\mathscr{P}, \leq)$. Indeed, there is the gap $[\vec{P}_1, \vec{P}_2]$, and then there is the linear order $(\mathcal{L}, \leq)$ in the interval $[\vec{P}_2, \vec{P}_3]$. Every other interval has the form $[P_1, P_2]$ where P_2 has height greater or equal to 4 and, thus, it is dense. Such intervals are not only dense but, in fact, universal. To prove this claim we shall proceed similarly to what we did in section 2.2, that is, we will construct an embedding from a partial order already known to be universal into the interval $[P_1, P_2]$. Due to its simplicity, we will use as a universal partial order the class of oriented paths itself.

Theorem 3.3.1 ([7]). *The homomorphism order of paths is universal.*

Our next result, Theorem 3.3.4 below, is the second main contribution of this work. It can be seen as a refinement of Theorem 3.3.1 in that not only the homomorphism order of paths is universal but we will show that in fact every dense interval is universal. We will start with some preparations to the proof.

For an arc $a = uv$ let us write $h(a) = u$ and $t(a) = v$ to denote the head and the tail of the arc a.

Claim 3.3.2. *Let P_1, P_2 be two paths and let $f : A(P_1) \to A(P_2)$ be a mapping. If for every pair of arcs $a, a' \in A(P_1)$ and every $z, z' \in \{h, t\}$,*

$$z(f(a)) = z'(f(a')) \quad \text{if } z(a) = z'(a')$$

then f induces a homomorphism $f' : P_1 \to P_2$.

Proof. Given a vertex $v \in P_1$ consider an arc $a \in A(P)$ such that $v = z(a)$ where $z \in \{h, t\}$. Let $f'(v) = z(f(a))$. To check that f' is a homomorphism it is enough to see that f' is well-defined since by definition f' preserves the arcs. Suppose that v is incident to two different arcs a, a', so $v = z(a) = z'(a')$ for some $z, z' \in \{h, t\}$. Then $f'(v) = z(f(a)) = z'(f(a'))$ is well-defined and $f' : P_1 \to P_2$ is a homomorphism. $\square$

Given a path $I \in \mathscr{P}$ we define

$$\Phi_I : \mathscr{P} \to \mathscr{P}$$

to be the map defined on arcs as $\Phi_I(qq') = I$ and, for a path $Q = (q_0, q_1, \ldots, q_n)$, $\Phi_I(Q) = \Phi_I(q_0q_1)^{\epsilon_0}\Phi_I(q_1q_2)^{\epsilon_1} \cdots \Phi_I(q_{n-1}q_n)^{\epsilon_{n-1}}$, where $\epsilon_i = 1$ if q_iq_{i+1} is an arc of Q and $\epsilon_i = -1$ if $q_{i+1}q_i$ is an arc of Q. In other words, $\Phi_I(Q)$ is obtained from Q by replacing every arc qq' by a copy of I by identifying q with $i(I)$ and q' with $t(I)$. Note that I acts as a gadget $I(a, b)$ where $a = i(I)$ and $b = t(I)$ as described in section 2.2.

Lemma 3.3.3. *Let I be a path. Suppose that, for every path Q, the following conditions hold:*

(i) $P_1 < \Phi_I(Q) < P_2$,

(ii) For every homomorphism $f : I \to \Phi_I(Q)$ there is an arc qq' of Q such that $f(I) \subseteq \Phi_I(qq')$.

(iii) For every homomorphism $g : I_1^{\epsilon_1} I_2^{\epsilon_2} \to \Phi_I(Q)$, where I_1, I_2 are two copies of I and $\epsilon_i \in \{-1, 1\}$, if $g(I_1^{\epsilon_1}) \subseteq \Phi_I(q_1 q_1')$ and $g(I_2^{\epsilon_2}) \subseteq \Phi_I(q_2 q_2')$ then

$$\begin{cases} q_1' = q_2 & \textit{if } \epsilon_1 = \epsilon_2 = 1 \\ q_1 = q_2' & \textit{if } \epsilon_1 = \epsilon_2 = -1 \\ q_1' = q_2' & \textit{if } \epsilon_1 = 1 \textit{ and } \epsilon_2 = -1 \\ q_1 = q_2 & \textit{if } \epsilon_1 = -1 \textit{ and } \epsilon_2 = 1. \end{cases}$$

Then Φ_I is a poset embedding $(\mathscr{P}, \leq)$ into the interval $[P_1, P_2]$.

Proof. The condition *(i)* ensures that $\Phi_I(\mathcal{P}) \subset [P_1, P_2]$.

Let Q_1, Q_2 be two paths and let $f : Q_1 \to Q_2$ be a homomorphism. Then f induces a homomorphism $g : \phi_I(Q_1) \to \phi_I(Q_2)$ by associating each arc $qq' \in Q_1$ (or Q_2) with $\Phi_I(qq') \subseteq \Phi_I(Q_1)$ (or $\Phi_I(Q_2)$), which makes the following diagram

$$\begin{array}{ccc} Q_1 & \xrightarrow{f} & Q_2 \\ \Big\downarrow{\scriptstyle \phi_I} & & \Big\downarrow{\scriptstyle \phi_I} \\ \phi_I(Q_1) & \xrightarrow{g} & \phi_I(Q_2). \end{array}$$

commutative.

Reciprocally, given a homomorphism $g : \phi_I(Q_1) \to \phi_I(Q_2)$, condition *(ii)* ensures that g induces a mapping $f : A(Q_1) \to A(Q_2)$ defined as $f(q_1 q_1') = q_2 q_2'$ for each arc $q_1 q_1' \in Q_1$ such that $g(\Phi_I(q_1 q_1')) \subseteq \Phi_I(q_2 q_2')$. Moreover, by condition *(iii)*, the resulting map f satisfies the condition of Claim 3.3.2 and hence, f is a homomorphism and the above diagram is commutative. Hence, $Q_1 \to Q_2$ if and only if $\Phi_I(Q_1) \to \Phi_I(Q_2)$, which shows that Φ_I is a poset embedding. □

Theorem 3.3.4. *Let P_1 and P_2 be two paths such that $P_1 < P_2$. If the height of P_2 is greater or equal to 4, then the interval $[P_1, P_2]$ is universal.*

The proof of the theorem consists of constructing an appropriate path I to ensure that $\Phi_I : \mathcal{P} \to [P_1, P_2]$ is a well-defined poset embedding by applying Lemma 3.3.3. Then, the universality of $[P_1, P_2]$ follows by Theorem 3.3.1.

Lemma 3.3.5. *Let P_1 and P_2 be two paths such that $P_1 < P_2$. If the height of P_2 is greater or equal to 4, then there exists a path P which is a core and satisfies $P_1 < P < P_2$, and a surjective homomorphism $h : P \to P_2$.*

Proof. By Theorem 3.2.3 there exists a path P such that $P_1 < P < P_2$. Assume P to be a core. Suppose that there is no surjective homomorphism $f : P \to P_2$, otherwise we are done. Then $f(P) \subsetneq P_2$ so at least one of $\{i(P_2), t(P_2)\}$ is not in $f(P)$. Suppose without loss of generality that $t(P_2) \notin f(P)$.

For an oriented path $P = (x_1, \ldots, x_n)$ we denote by $d(x_i, x_j) = |j - i|$ the distance form x_i to x_j in the underlying undirected path.

Claim 3.3.6. *Suppose that* $\max_{f\in Hom(P,P_2)}\{d(t(f(P)), t(P_2))\} > 0$. *Then there is a core* $P' \subset P_2$ *such that* $P < P' < P_2$ *and*

$$\max_{f\in Hom(P',P_2)}\{d(t(f(P')), t(P_2))\} < \max_{f\in Hom(P,P_2)}\{d(t(f(P)), t(P_2))\}.$$

Proof. Let f be a homomorphism $f : P \to P_2$ which minimises the distance $d(i(P_2), t(f(P)))$. Let A and B be the paths from $i(P_2)$ to $t(f(P))$ and from $t(f(P))$ to $t(P_2)$ respectively. Note that A and B are non empty paths, $f(P) \subseteq A$ and $P_2 = AB$. Let B^* be the subpath of P_2 consisting of B and the last arc of A. Consider the concatenation path AZB where Z is a zig-zag of level equal to the last arc of A and even length greater than $\max\{|P|, |P_2|\}$. We have that $AZ \to A$ and $ZB \to B^*$.

We now show that $P < AZB < P_2$. It is clear that $P \to AZB \to P_2$. Suppose that $P_2 \to AZB$. Since $|Z| > |P_2|$, either $P_2 \to AZ \to A$ or $P_2 \to ZB \to B^*$, and both cases are a contradiction since P_2 is a core. Thus, $P_2 \nrightarrow AZB$. Suppose now that there exists a homomorphism $g : AZB \to P$. We then have $P \to f(P) \to P$ which must be equal to the identity mapping by Lemma 3.1.11. Thus, $f(P)$ is isomorphic to P so we can consider the homomorphism $g' : AZB \to f(P)$. Note that the restriction $g'|_{f(P)}$ must also be the identity mapping. Since $f(P)$ ends with two arcs in the same direction because it is a core, it follows that g' must collapse the zig-zag Z to the last arc of $f(P)$. Then g' induces a homomorphism $P_2 = AB \to f(P)$ contradicting that $P_2 \nrightarrow P$.

Let C be the core of AZB. We show now that $C \nrightarrow A$, or equivalently, $C \nsubseteq A$.

Suppose that $C \subseteq A$. Let $\rho : AZB \to C$ be a homomorphism. If the last arc of A is not in C, then we can consider $\rho \circ f : P \to C$ as a homomorphism $P \to P_2$ such that $t((\rho \circ f)(P))$ is at the left of $t(f(P))$, a contradiction. Otherwise, the last arc of A is the last arc of C. Note that the restriction $\rho|_C$ must be the identity mapping since C is a core. Since C ends with two arcs in the same direction, it follows that ρ must collapse the zig-zag Z to the last arc of C. Then ρ induces a homomorphism $AB \to C$ contradicting that $P_2 \nrightarrow A$ since it is a core. Hence, $C \nsubseteq A$ which implies that $C \nrightarrow A$ and we can take $P' = C$ to prove the claim. □

By Claim 3.3.6, since the paths are finite, we may assume that $P \subset P_2$ and $t(f(P)) = t(P_2)$ for every homomorphism $f : P \to P_2$.

In particular, by considering the inclusion mapping $P \hookrightarrow P_2$, we have $t(P) = t(P_2)$. Since $P_2 \nrightarrow P$ we have $P \neq P_2$. Let A' be the path from $i(P_2)$ to $i(P)$. Note that A' is a non empty path and $P_2 = A'P$. Let A^* be the subpath of P_2 consisting of A' and the first arc of P. Consider the concatenation path $A'Z'P$ where Z' is a zig-zag of level equal to the first arc of P and even length greater than $|P_2|$.

Let C' be the core of $A'Z'P$ and let $\rho' : A'Z'P \to C'$ be a homomorphism. Suppose that $Z \cap C' = \emptyset$, so that either $C' \subseteq A^*$ or $C' \subseteq P$. The first option is not possible since $P \to C' \hookrightarrow A^* \subset P_2$ contradicts $t(C') = t(P_2)$. If $C' \subseteq P$, then $C' = P$ because both paths are cores. Then the restriction $\rho'|_P$ must be the identity mapping. Since P starts with two arcs in the same direction, it follows that ρ' must collapse the zig-zag Z' to the first arc of P which induces a homomorphism $P_2 = A'P \to P$ contradicting that $P_2 \nrightarrow P$. Hence $Z' \subset C'$ and, as consequence, $C' = A'Z'P$. Observe that we can obtain a surjective homomorphism $C' = A'Z'P \to P_2 = A'P$ by collapsing the zig-zag Z' to the first arc of P. Finally, since $P_2 \nrightarrow C'$ and $P \to C'$, we have $P_1 < C' < P_2$ which concludes the proof. □

Given $P_1 < P_2$, Theorem 3.2.3 and Lemma 3.3.5 imply that there exists a path P which is a core and satisfies $P_1 < P < P_2$, and a surjective homomorphism

$$h : P \to P_2.$$

Observe that h cannot be a one to one mapping, for otherwise h^{-1} would be a homomorphism from P_2 to P_1, contrary to our assumption that $P_2 \nrightarrow P_1$. Thus, there must exist two different vertices $v_1, v_2 \in P$ such that $h(v_1) = h(v_2)$. Actually, it is not difficult to see that there exist a pair of vertices $v_1, v_2 \in P$ with a common neighbour v_0 such that $h(v_1) = h(v_2)$, so let us assume that this is the case. Then we have $l(v_1) = l(v_2)$, and either both v_1v_0 and v_2v_0 are arcs of P or both v_0v_1 and v_0v_2 are arcs of P. By inverting all orientations if necessary we may assume that v_1v_0 and v_2v_0 are arcs of P. Suppose without loss of generality that v_1 is at the left of v_2 in the sequence of P. Let A, B, C be the path from $i(P)$ to v_1, from v_1 to v_2 and from v_2 to $t(P)$ respectively. Then we have that $P = ABC$. Note that A and C must be non empty otherwise P would not be a core. Note also that B consists only on the two arcs v_1v_0 and v_2v_0. Let v_1' be the vertex in A adjacent to v_1 and let v_2' be the vertex in C adjacent to v_2.

Let a_1 be the arc between v_1 and v_1' and let a_2 be the arc between v_2 and v_2'. The construction of the path I will depend on the orientation of the arcs a_1 and a_2. However, in all cases it will have the form

$$I = Z_1 A^{-1} ABCC^{-1} Z_2 XX^{-1}, \tag{3.1}$$

where Z_1 and Z_2 are zig-zags of even length greater than $|P|$ and level equal to $l(a_1)$ and $l(a_2)$ respectively, and X is a (possibly empty) path such that $X \subseteq C$ and $i(X) = i(C)$. Observe that there is a homomorphism

$$\rho : I \to ABC = P$$

that collapses the zig-zags Z_1 and Z_2 into the arcs a_1 and a_2 and maps the rest of vertices in I to their corresponding vertices in ABC. Then the mapping

$$\rho' : \Phi_I(Q) \to P_2$$

defined as $\rho'(v) = (h \circ \rho)(v)$ is a well-defined homomorphism since $h(v_1) = h(v_2)$.

Lemma 3.3.7. *Let* $I = Z_1A^{-1}ABCC^{-1}Z_2XX^{-1}$ *as defined in* (3.1) *and suppose that* $P \nrightarrow Z_2XX^{-1}Z_1$*. Then, for every path* Q*,*

(i) $P_1 < \Phi_I(Q) < P_2$*,*

(ii) For every homomorphism $f : I \to \Phi_I(Q)$ *there is an arc* qq' *of* Q *such that* $f(I) \subseteq \Phi_I(qq')$*. Moreover,* $f|_P(ABC) = ABC \subset \Phi_I(qq')$ *and* $f|_P$ *is the identity mapping.*

Proof. It is clear that $P_1 < \Phi_I(Q)$ since $P \subset I \subseteq \Phi_I(Q)$. We know also that $\Phi_I(Q) \to P_2$ via the homomorphism ρ'. Suppose that $g : P_2 \to \Phi_I(Q)$ is a homomorphism. Since Z_1 and Z_2 are larger than $|P|$ and, in particular, than $|P_2|$ (because $h : P \to P_2$ is surjective) then $g(P_2)$ is either contained in some copy of *(a)* $Z_1A^{-1}ABCC^{-1}Z_2$, *(b)* $Z_2XX^{-1}Z_2$ or *(c)* $Z_2XX^{-1}Z_1$. Since $Z_1A^{-1}ABCC^{-1}Z_2 \to P$ and $Z_2XX^{-1}Z_2 \to C$ but $P_2 \nrightarrow P$ then *(a)* and *(b)* are not possible. Moreover, since we are assuming that $P \nrightarrow Z_2XX^{-1}Z_1$ and $P \to P_2$, then *(c)* is not possible either. Thus, $P_2 \nrightarrow \Phi_I(Q)$ and *(i)* holds.

Let $f : I \to \Phi_I(Q)$ be a homomorphism. Consider the restriction $f|_P : P \to \Phi_I(Q)$. Since Z_1 and Z_2 are longer than $|P|$ then $f|_P(P)$ is either contained in some copy of *(a)* $Z_1A^{-1}ABCC^{-1}Z_2$, *(b)* $Z_2XX^{-1}Z_2$ or *(c)* $Z_2XX^{-1}Z_1$. By assumption *(c)* is not possible. Since $Z_2XX^{-1}Z_2 \to C \subset P$ but P is a core, *(b)* can not hold either. Thus, $f|_P(P) \subset Z_1A^{-1}ABCC^{-1}Z_2 \subseteq \Phi_I(qq')$ for some $qq' \in A(Q_2)$. Finally, note that the composition $(\rho \circ f|_P) : P \to P$ must be the identity mapping since P is rigid by Lemma 3.1.11 and, thus, $f_P(P)$ must be isomorphic to P. As consequence, the only possibility is that $f|_P(P) = ABC \subset \Phi_I(qq')$ and f_P is the identity mapping. $\square$

Observe that Lemma 3.3.7 guarantees the first two conditions *(i)* and *(ii)* of Lemma 3.3.3. So given a path I of the form $I = Z_1A^{-1}ABCC^{-1}Z_2XX^{-1}$ as described in (3.1), showing that $P \nrightarrow Z_2XX^{-1}Z_1$ and condition *(iii)* of Lemma 3.3.3 holds is enough to claim that $\Phi_I : \mathcal{P} \to [P_1, P_2]$ is a poset embedding, which proves Theorem 3.3.4.

Proof of Theorem 3.3.4. We shall consider four different cases according to the directions of the arcs a_1 and a_2 in P. As mentioned above, in all cases the path I will have the form $I = Z_1A^{-1}ABCC^{-1}Z_2XX^{-1}$, so it is enough to check that for each case $P \nrightarrow Z_2XX^{-1}Z_1$ and that condition *(iii)* of Lemma 3.3.3 holds.

Let I_1, I_2 be two copies of I and denote by L_1 and L_2 the copies of P in I_1 and I_2 respectively. By Lemma 3.3.7, $g(I_1) \subseteq \Phi_I(q_1q_1')$ and $g(I_2) \subseteq \Phi_I(q_2q_2')$ for some arcs $q_1q_1', q_2q_2' \in Q$. Let us write

$$I_1' = \Phi_I(q_1q_1') \text{ and } I_2' = \Phi_I(q_2q_2'),$$

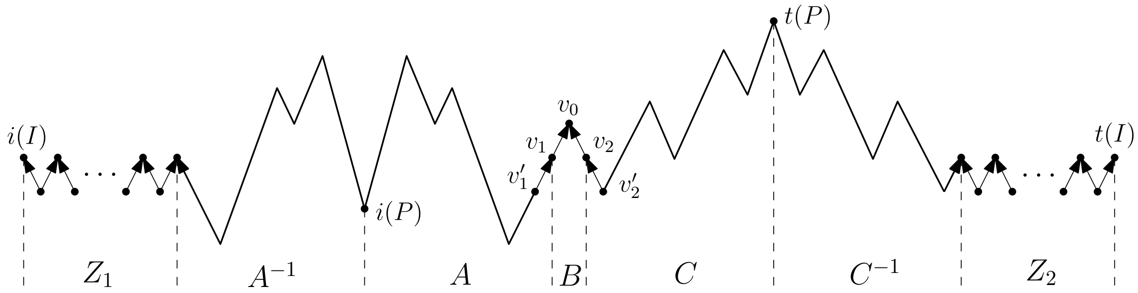

Figure 3.6: Example of how path I might look in Case 1.

and denote by L_1' and L_2' the copies of P in I_1' and I_2' respectively. By Lemma 3.3.7 (ii) we have

$$g(L_1) = L_1' \text{ and } g(L_2) = L_2',$$

and the restrictions $g|_{L_i}$ are the identity mapping. In particular, $g(i(L_1)) = i(L_1')$, $g(t(L_1)) = t(L_1')$, $g(i(L_2)) = i(L_2')$ and $g(t(L_2)) = t(L_2')$.

Let $k = l(v_1) = l(v_2) = l(v_1v_0) = l(v_2v_0)$.

Case 1: $a_1 = v_1'v_1$ and $a_2 = v_2'v_2$ are arcs of P.

Let

$$I = Z_1A^{-1}ABCC^{-1}Z_2,$$

where Z_1 and Z_2 are zig-zags of the same even length greater than $|P|$ and level equal to $k-1 = l(a_1) = l(a_2)$. In this case X is the empty path and it is clear that $P \nrightarrow Z_2Z_1$. See Figure 3.6. It only remains to show that condition (iii) in Lemma 3.3.3 holds to prove the Theorem.

Consider a homomorphism

$$g : I_1^{\epsilon_1}I_2^{\epsilon_2} \to \Phi_I(Q)$$

Since $I_1^{\epsilon_1}$ and $I_2^{\epsilon_2}$ intersect in a vertex, then I_1' and I_2' must also intersect. Therefore, one of the following holds: either $I_1' = I_2'$, or $I_1' \neq I_2'$ and (a) $t(I_1) = i(I_2)$, or (b) $t(I_1) = t(I_2)$, or (c) $i(I_1) = i(I_2)$, or (d) $i(I_1) = t(I_2)$. See Figure 3.7.

Suppose first that $\epsilon_1 = \epsilon_2 = 1$. Then, $t(L_1)$ is joined to $i(L_2)$ by the path

$$C^{-1}Z_2Z_1A^{-1}.$$

If $I_1' = I_2'$, then $t(L_1')$ is joined to $i(L_2')$ by the path $C^{-1}B^{-1}A^{-1}$, but since B is a zig-zag of level k while Z_2Z_1 is a zig-zag of level $k-1$, then $g(Z_2Z_1) \cap B = \emptyset$. This contradicts that g is a homomorphism satisfying $g(t(L_1)) = t(L_1')$ and $g(i(L_2)) = i(L_2')$. On the other hand, if $I_1' \neq I_2'$, then we can discard (b), (c) and (d) since, in such cases, the distance between $t(L_1')$ and $i(L_2')$ would be greater than the distance

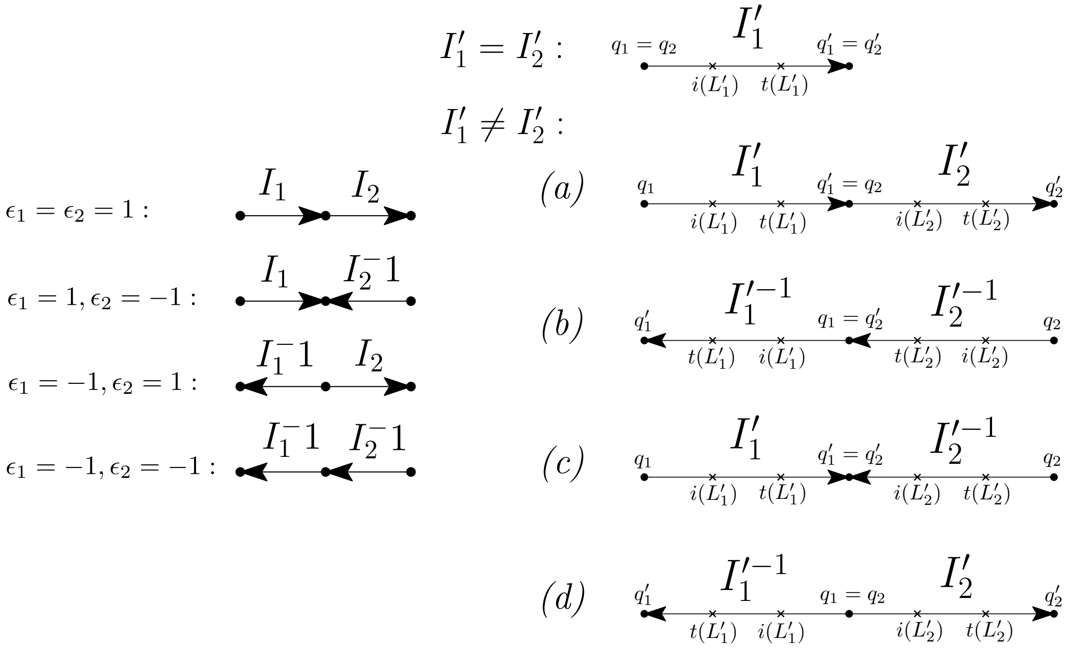

Figure 3.7: Possible configurations of $I_1^{\epsilon_1} I_2^{\epsilon_2}$ and I_1' and I_2'. The arrow indicates the direction of the entire path I.

between $t(L_1)$ and $i(L_2)$. See Figure 3.7. Thus, (a) must hold and condition (iii) of Lemma 3.3.3 holds in this case.

Suppose now that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. If $I_1' \neq I_2'$, we can discard (a), (b) and (d) since in those cases the distance between $t(L_1')$ and $t(L_2')$ would be greater than the distance between $t(L_1)$ and $t(L_2)$. Thus, either $I_1' = I_2'$ or (c) holds, both cases yielding condition (iii) of Lemma 3.3.3.

Finally, the case $\epsilon_1 = \epsilon_2 = -1$ and the case $\epsilon_1 = -1$ and $\epsilon_2 = 1$ are symmetric to the previous cases, completing the proof of the Theorem in Case 1.

Case 2: $a_1 = v_1'v_1$ and $a_2 = v_2v_2'$ are arcs of P.

In this case $l(a_1) = k - 1$ and $l(a_2) = k$. Let $w \in C$ be the closest vertex to v_2' with level different to k and $k + 1$. Since P is a core, one of the following two cases must occur.

Case 2.a: $l(w) = k + 2$.

Let D be the path from v_2 to w (see Figure 3.8). So D satisfies that $D \subseteq C$ and $i(D) = i(C) = v_2$. The path I is now defined as

$$I = Z_1 A^{-1} ABCC^{-1} Z_2 DD^{-1},$$

where Z_2 is a zig-zag of even length greater than $|P|$ and level equal to k, and Z_1

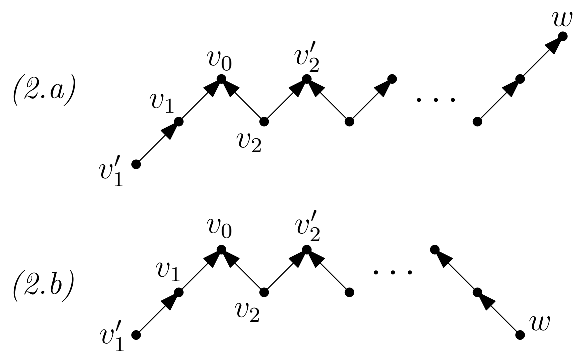

Figure 3.8:

is a zig-zag of length equal to $|Z_2| + |DD^{-1}|$ and level $k-1$. First, observe that $P \nrightarrow Z_2DD^{-1}Z_1$ since $Z_2DD^{-1}Z_1$ is a path of height 3 and P is a path of height greater or equal to 4. Again it only remains to show that condition (iii) in Lemma 3.3.3 holds to prove the Theorem.

Consider a homomorphism

$$g : I_1^{\epsilon_1} I_2^{\epsilon_2} \to \Phi_I(Q)$$

Recall that we denote by I_1' and I_2' the copies of I in $\phi_I(Q)$ which are images of arcs in Q and contain $g(I_1)$ and $g(I_2)$ respectively. One of the following holds: either $I_1' = I_2'$, or $I_1' \neq I_2'$ and (a) $t(I_1') = i(I_2')$, or (b) $t(I_1') = t(I_2')$, or (c) $i(I_1') = i(I_2')$, or (d) $i(I_1') = t(I_2')$. See Figure 3.7.

Suppose that $\epsilon_1 = \epsilon_2 = 1$. Then, $t(L_1)$ is joined to $i(L_2)$ by the path

$$C^{-1}Z_2DD^{-1}Z_1A^{-1}.$$

If $I_1' = I_2'$ then $t(L_1')$ is joined to $i(L_2')$ by the path $C^{-1}B^{-1}A^{-1}$. Since Z_2 is a zig-zag of level k, D consists of a zig-zag of level k followed by an arc of level $k+1$, and A^{-1} starts with an arc of level $k-1$, we have $g(C^{-1}Z_2D) \subseteq C^{-1}B^{-1}$, or equivalently $g(C^{-1}Z_2D) \cap A^{-1} = \emptyset$. It follows that, if $d \in DD^{-1}$ denotes the middle vertex in DD^{-1}, then $g(d) \in C^{-1}$ since d is a vertex of level $k+2$. It follows that $g(D^{-1}Z_1) \subseteq C^{-1}$ since Z_1 has level $k-1$ and B^{-1} has level k. Hence, $g(Z_1) \cap B^{-1}A^{-1} = \emptyset$ contradicting that g is a homomorphism satisfying $g(t(L_1)) = t(L_1')$ and $g(i(L_2)) = i(L_2')$. It follows that $I_1' \neq I_2'$. Then we can discard (b), (c) and (d) since, in such cases, the distance between $g(t(L_1))$ and $g(i(L_2))$ would be greater than the distance between $t(L_1)$ and $i(L_2)$. Thus, (a) holds yielding condition (iii) of Lemma 3.3.3.

Suppose now that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. If $I_1' \neq I_2'$, we can discard (a), (b) and (d) since in these cases the distance between $g(t(L_1))$ and $g(t(L_2))$ would be greater than the distance between $t(L_1)$ and $t(L_2)$. Thus, either $I'_1 = I'_2$ or (c) holds, both cases yielding condition (iii) of Lemma 3.3.3 holds.

Finally, the case $\epsilon_1 = \epsilon_2 = -1$ and the case $\epsilon_1 = -1$ and $\epsilon_2 = 1$ are symmetric to the previous cases. Hence, condition (*iii*) holds.

Case 2.b: $l(w) = k - 1$.

Let
$$I = Z_1 A^{-1} ABCC^{-1} Z_2 CC^{-1}$$
where Z_2 is a zig-zag of even length greater than $|P|$ and level equal to k, and Z_1 is a zig-zag of length equal to $|Z_2| + |CC^{-1}|$ and level $k-1$. Observe that $C^{-1}Z_1 \to C^{-1}$ since C starts with an arc of level k followed by an arc of level $k-1$ (see Figure 3.8). Then, $P \nrightarrow Z_2CC^{-1}Z_1$ because otherwise $P \to Z_2CC^{-1}Z_1 \to C$, but P is a core.

Let us show that condition (iii) of Lemma 3.3.3 holds in this case as well. Consider a homomorphism
$$g : I_1^{\epsilon_1} I_2^{\epsilon_2} \to \Phi_I(Q).$$
As before we have either $I'_1 = I'_2$, or $I'_1 \neq I'_2$ and (a) $t(I'_1) = i(I'_2)$, or (b) $t(I'_1) = t(I'_2)$, or (c) $i(I'_1) = i(I'_2)$, or (d) $i(I'_1) = t(I'_2)$. See Figure 3.7.

Suppose first that $\epsilon_1 = \epsilon_2 = 1$. Then, $t(L_1)$ is joined to $i(L_2)$ by the path
$$C^{-1} Z_2 CC^{-1} Z_1 A^{-1}.$$

Assume that $I'_1 = I'_2$. Then $g(t(L_1))$ is joined to $g(i(L_2))$ by the path $C^{-1}B^{-1}A^{-1}$.

Claim 3.3.8. *Let c be the middle vertex of CC^{-1}. Then either $g(c) = t(L'_1)$ or $g(c) \notin L'_1$.*

Proof. If $g(c) \in L'_1$ but $g(c) \neq t(L'_1)$ then there is a vertex $s \in CC^{-1}$ such that $g(s) = s \in C^{-1} \subset L'^{-1}_1 = C^{-1}B^{-1}A^{-1}$. Let S be the path from s to c in CC^{-1}. By considering S as a subpath of C and $s \in C \subset P$, we can define an automorphism $P \to P$ equal to the identity mapping on the vertices from $i(P)$ to s and equal to $g|_S$ on the vertices from s to $t(P)$. However, such automorphism would not be equal to the identity mapping since $g(c) \neq t(L'_1)$, contradicting that P is rigid. □

By Claim 3.3.8, we have $g(Z_1) \cap B^{-1}A^{-1} = \emptyset$ since Z_1 has level $k-1$ while B has level k. This contradicts that g is a homomorphism satisfying $g(t(L_1)) = t(L'_1)$ and $g(i(L_2)) = i(L'_2)$. Hence, we have $I'_1 \neq I'_2$. Then we can discard (b), (c) and (d) since, in such cases, the distance between $g(t(L_1))$ and $g(i(L_2))$ would be greater than then distance between $t(L_1)$ and $i(L_2)$. Thus, condition (iii) of Lemma 3.3.3 holds in this case.

Suppose now that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. If $I'_1 \neq I'_2$, we can discard (a), (b) and (d) since in these cases the distance between $g(t(L_1))$ and $g(t(L_2))$ would be greater than the distance between $t(L_1)$ and $t(L_2)$. Thus, either $I'_1 = I'_2$ or (c) holds, both cases yielding condition (iii) of Lemma 3.3.3 holds.

Finally, the case $\epsilon_1 = \epsilon_2 = -1$ and the case $\epsilon_1 = -1$ and $\epsilon_2 = 1$ are symmetric to the previous cases. This completes the proof in this case.

Case 3: $a_1 = v_1v'_1$ and $a_2 = v'_2v_2$ are arcs of P.

In this case P^{-1} is as in Case 2 and the arguments in that case apply.

Case 4: $a_1 = v_1v'_1$ and $a_2 = v_2v'_2$ are arcs of P.

Recall that $h : P \to P_2$ is a surjective homomorphism such that $h(v_1) = h(v_2)$. Let $w \in C$ be the closest vertex to v'_2 with level different to k and $k + 1$.

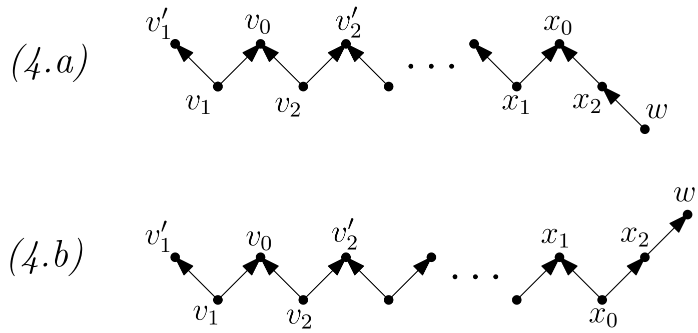

Figure 3.9:

Case 4.a: $l(w) = k - 1$.

Let P' be the path obtained from P by removing the vertex v_0 and identifying the vertices v_1 and v_2. Note that $P' = AC$. It is clear that $h : P \to P' \to P_2$. Let x_1, x_0, x_2 be the vertices from Figure 3.9. Let X be the path obtained from P by removing the vertex x_0 and identifying the vertices x_1 and x_2. Then X is isomorphic to P' so there exists a surjective homomorphism $h' : P \to X \to P_2$ such that $h'(x_1) = h'(x_2)$. Thus, X is equivalent to P in Case 3.

Case 4.b: $l(w) = k + 2$.

Observe that if $h(v_0) \neq h(v'_1)$ and $h(v'_2)$, then $h(v'_1) = h(v'_2)$ and we can modify h to map the vertex v_0 to $h(v'_1)$ and still obtain a homomorphism. So suppose without loss of generality that $h(v_0) = h(v'_1)$. Let P' be the path obtained from P by removing the vertex v_1 and identifying the vertices v_0 and v'_1. It is clear that $P \to P' \to P_2$. Let x_1, x_0, x_2 be the vertices from Figure 3.9. Let X be the path obtained from P by removing the vertex x_0 and identifying the vertices x_1

and x_2. Then X is isomorphic to P' so there exists a surjective homomorphism $h' : P \to X \to P_2$ such that $h'(x_1) = h'(x_2)$. Thus, X is equivalent to P in Case 2. $\square$

Corollary 3.3.9. *The class of paths of height greater or equal to 4 has the fractal property.*

3.4 Density for Trees

In the previous section we have shown that intervals of the form $[T_1, P_2]$, where the core of P_2 is a path of height greater or equal to 4, are dense in the homomorphism order of trees. That is, there exists a tree T such that $T_1 < T < P_2$. We are interested in extending such result to intervals of the form $[T_1, T_2]$ where T_2 is any oriented tree of height greater or equal to 4. However, the proof of Theorem 3.2.1 uses the fact that P_2 is a path and there is no obvious way to generalise it. For this reason, instead of extending the proof we look for the opposite case, which are intervals $[T_1, T_2]$ where T_2 is a tree whose core is not a path. Surprisingly, we shall show a density theorem for these intervals with an entirely different proof. Theorem 3.4.2 below is another main contribution in this work.

We say that a tree is *proper* if its core is not a path. Thus, a tree is proper if its core has at least one vertex of degree at least three. Observe that, by propositions 3.1.6 and 3.1.8, every proper tree has height greater or equal to 4.

In order to prove a density theorem we shall construct a tree $\mathcal{D}_n(T_2)$ from a given proper tree T_2 which will satisfy $T_1 < \mathcal{D}_n(T_2) < T_2$ for every tree $T_1 < T_2$.

Construction of $\mathcal{D}_n(T_2)$:

Let T_2 be the core of a proper tree. Then there exists a vertex $x \in V(T_2)$ such that x is adjacent to at least three different vertices, call them u, v, w. Without loss of generality we shall assume that ux and wx or xu and xu are arcs of T_2. In fact, we can assume that ux and wx are arcs; for the other case we would applied the same construction but changing the direction of all arcs appearing in it. Let $X' \subseteq T_2$ be the set of vertices, different from u and w, which are adjacent to x. Note that X' is not empty since $v \in X'$. Let $X \subset T_2$ be the plank from x to X'. Let $U' = P(x, \{u\})$ and let $W' = P(x, \{w\})$. Let U and W be the tree obtained from U' and W' by removing the vertex x respectively. Observe that U is not empty since otherwise there exists an automorphism $T_2 \to T_2$ which maps the arc ux to the arc wx implying that T_2 is not a core. By the same argument we see that W is also not empty. Note that $U \sqcup X \sqcup W \sqcup \{ux, wx\} = T_2$. See Figure 3.10.

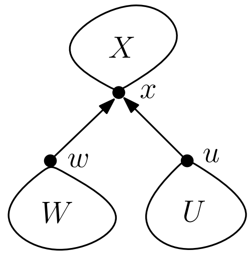

Figure 3.10: Tree T_2

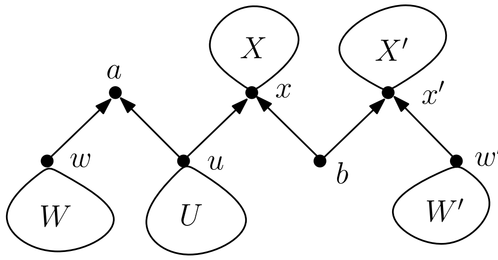

Figure 3.11: Tree $\mathcal{D}_1(T)$

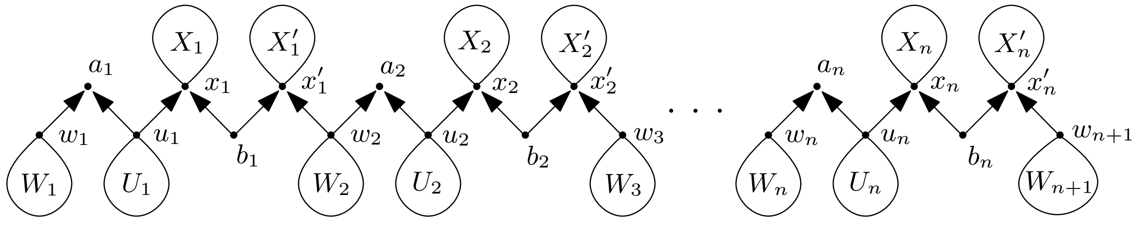

Figure 3.12: Tree $\mathcal{D}_n(T_2)$. Observe the enumeration of the vertices and planks of each tree $\mathcal{D}_1(T)$

Now, let $\mathcal{D}_1(T_2)$ be the tree from Figure 3.11, where W and W' are copies of the plank $W \subset T_2$, U is a copy of $U \subset T_2$, and X and X' are copies of $X \subset T_2$.

Finally, let $\mathcal{D}_n(T_2)$ be a tree consisting in n consecutive trees $\mathcal{D}_1(T_2)$ whose planks W' are identified with the planks W of the following trees as shown in Figure 3.12. We shall refer to the vertices $w_i, a_i, u_i, x_i, b_i, x'_i \in \mathcal{D}_n(T_2)$ for $i = 1, \ldots, n$ as *labelled vertices*. Note that $\mathcal{D}_n(T_2)$ is a proper tree for any $n > 0$.

Lemma 3.4.1. *Let T_1 and T_2 be finite oriented trees such that T_2 is a proper tree and $T_2 \nrightarrow T_1$. If there exists a homomorphism $f : \mathcal{D}_n(T_2) \to T_1$, then every labelled vertex of $\mathcal{D}_n(T_2)$ is mapped to a different vertex of T_1.*

Proof. Assume that T_2 is a core and consider a homomorphism $f : \mathcal{D}_n(T_2) \to T_1$. Observe that two consecutive labelled vertices can not be mapped via f to the same vertex since it would imply that T_1 contains a loop. Now, observe that if any pair of labelled vertices of distance two are mapped to the same vertex, it will induce a homomorphism $T_2 \to T_1$. This follows from the construction of $\mathcal{D}_n(T_2)$. See Figure 3.12. Finally, if two labelled vertices of distance greater or equal to three are mapped to the same vertex, it would imply that T_1 contains a cycle since every pair of labelled vertices of distance less than three are mapped to different vertices, but T_1 is a tree. We conclude that every labelled vertex has to be mapped to a different vertex of T_1. $\square$

Theorem 3.4.2. *Let T_1 and T_2 be two oriented trees satisfying $T_1 < T_2$. If T_2 is a proper tree, then there exists a tree T such that $T_1 < T < T_2$.*

Proof. Assume that T_2 is a core. Let $n > |T_1|$ and consider the tree $\mathcal{D}_n(T_2)$ constructed from T_2. We claim that $T_1 < T_1 + \mathcal{D}_n(T_2) < T_2$.

Observe that there exists a homomorphism $h : \mathcal{D}_n(T_2) \to T_2$ which maps each vertex of $\mathcal{D}_n(T_2)$ to its corresponding vertex in T_2 (mapping the vertices a_i to x and the vertices b_i to either w or u for $i = 1, \ldots, n$). Suppose there exists a homomorphism $g : T_2 \to \mathcal{D}_n(T_2)$. Recall from Proposition 1.3.7 that since the core of a tree is rigid, the plank U has to be mapped to some plank $U_i \subset \mathcal{D}_n(T_2)$ mapping all vertices of U to its corresponding vertices of U_i. Otherwise, $(h \circ g)(U)$ would be different from the identity on U, and hence, the composition $h \circ g : T_2 \to T_2$ would be a homomorphism different from the identity, contradicting Lemma 3.1.11. The same happens to the planks X and W; W has to be mapped to some $W_j \subset \mathcal{D}_n(T_2)$ and X has to be mapped to either some X_k or $X'_k \subset \mathcal{D}_n(T_2)$. However, there are not three consecutive vertices u_i, x_k, w_j or u_i, x'_k, w_j in $\mathcal{D}_n(T_2)$. Then $\mathcal{D}_n(T_2) < T_2$.

Suppose there exists a homomorphism $f : \mathcal{D}_n(T_2) \to T_1$. By Lemma 3.4.1 we know that f must map every labelled vertex of $\mathcal{D}_n(T_2)$ to a different vertex of T_1. However, since $n > |T_1|$, the number of labelled vertices of $\mathcal{D}_n(T_2)$ is greater than the number of vertices of T_1. Then $T_1 < T_1 + \mathcal{D}_n(T_2)$.

We end by considering the tree T consisting in the joining of T_1 and $\mathcal{D}_n(T_2)$ by a proper and long enough zig-zag. Then, by Lemma 2.1.4, $T_1 < T < T_2$. □

We are now able to give a density theorem for all oriented trees.

Theorem 3.4.3. *Let T_1 and T_2 be two oriented trees satisfying $T_1 < T_2$. If the height of T_2 is greater or equal to 4, then there exists a tree T such that $T_1 < T < T_2$.*

Proof. We split the proof in two cases: if the core of T_2 is a path then the result follows by Theorem 3.2.1, else if T_2 is a proper tree then Theorem 3.4.2 concludes the proof. □

3.5 Trees and the Fractal Property

We have seen that the class of trees of height greater or equal to 4 is dense. Moreover, as it happens in the class of paths, we will see that every interval is not only dense but universal. In other words, the class of trees of height greater or equal to 4 has the fractal property. To prove this we shall again split the problem into two cases: whether the upper tree of the interval is a path or a proper tree. If it is a path the case will be solved by Theorem 3.3.4. So in this section we will focus on proving the universality of intervals $[T_1, T_2]$ where T_2 is a proper tree.

Let $I(a,b)$ be a tree with two distinguished vertices. Given two copies $I_1(a_1,b_1)$, $I_2(a_2,b_2)$ of $I(a,b)$, let us write $I_1^{\epsilon_1}I_2^{\epsilon_2}$ to denote the tree obtained by identifying the vertex a_1 if $\epsilon_1=-1$, or the vertex b_1 if $\epsilon_1=1$, with the vertex a_2 if $\epsilon_2=1$, or the vertex b_2 if $\epsilon_2=-1$. As a comparison to paths, $I(a,b)$ can be seen as a tree with a "direction" where a is the initial vertex and b is the terminal vertex, I^{-1} is the inverse tree $I(b,a)$, and I_1I_2 is the concatenation tree obtained from identifying b_1 with a_2.

Given a tree $I\in\mathscr{T}$ we define

$$\Phi_I:\mathscr{P}\to\mathscr{T}$$

to be the map defined on arcs as $\Phi_I(qq')=I$ and, for a path $Q=(q_0,q_1,\dots,q_n)$, $\Phi_I(Q)=\Phi_I(q_0q_1)^{\epsilon_0}\Phi_I(q_1q_2)^{\epsilon_1}\cdots\Phi_I(q_{n-1}q_n)^{\epsilon_{n-1}}$, where $\epsilon_i=1$ if q_iq_{i+1} is an arc of Q and $\epsilon_i=-1$ if $q_{i+1}q_i$ is an arc of Q. In other words, $\Phi_I(Q)$ is obtained from Q by replacing every arc qq' by a copy of $I(a,b)$ by identifying q with a and q' with b. Again, $I(a,b)$ acts as the gadget described in section 2.2.

Lemma 3.5.1. *Let T_1,T_2 be two trees such that $T_1<T_2$. Let $I(a,b)$ be a tree. Suppose that, for every path Q, the following conditions hold:*

(i) $T_1<\Phi_I(Q)<T_2$,

(ii) For every homomorphism $f:I\to\Phi_I(Q)$ there is an arc qq' of Q such that $f(I)\subseteq\Phi_I(qq')$.

(iii) For every homomorphism $g:I_1^{\epsilon_1}I_2^{\epsilon_2}\to\Phi_I(Q)$, where I_1,I_2 are two copies of I and $\epsilon_i\in\{-1,1\}$, if $g(I_1^{\epsilon_1})\subseteq\Phi_I(q_1q_1')$ and $g(I_2^{\epsilon_2})\subseteq\Phi_I(q_2q_2')$ then

$$\begin{cases} q_1'=q_2 & \textit{if } \epsilon_1=\epsilon_2=1\\ q_1=q_2' & \textit{if } \epsilon_1=\epsilon_2=-1\\ q_1'=q_2' & \textit{if } \epsilon_1=1 \textit{ and } \epsilon_2=-1\\ q_1=q_2 & \textit{if } \epsilon_1=-1 \textit{ and } \epsilon_2=1.\end{cases}$$

Then Φ_I is a poset embedding from $(\mathscr{P},\leq)$ into the interval $[T_1,T_2]$.

Proof. The proof is analogous to Lemma 3.3.3. □

Theorem 3.5.2. *Let T_1 and T_2 be two oriented trees satisfying $T_1<T_2$. If T_2 is a proper tree, then the interval $[T_1,T_2]$ is universal.*

Proof. Assume T_2 to be a core. For enough large n, it can be checked that the core of $\mathcal{D}_n(T_2)$ must have more than $|T_1|$ labelled vertices by applying Lemma 3.4.1. Let D be the core of $\mathcal{D}_n(T_2)$ and let d_1 and d_2 be its initial and ending labelled vertex. Consider the tree obtained from D by adding two new vertices d_1' and d_2', and joining them to d_1 and d_2 by a zig-zag Z_1 and Z_2 respectively, of level equal to $l(wx)=l(ux)$

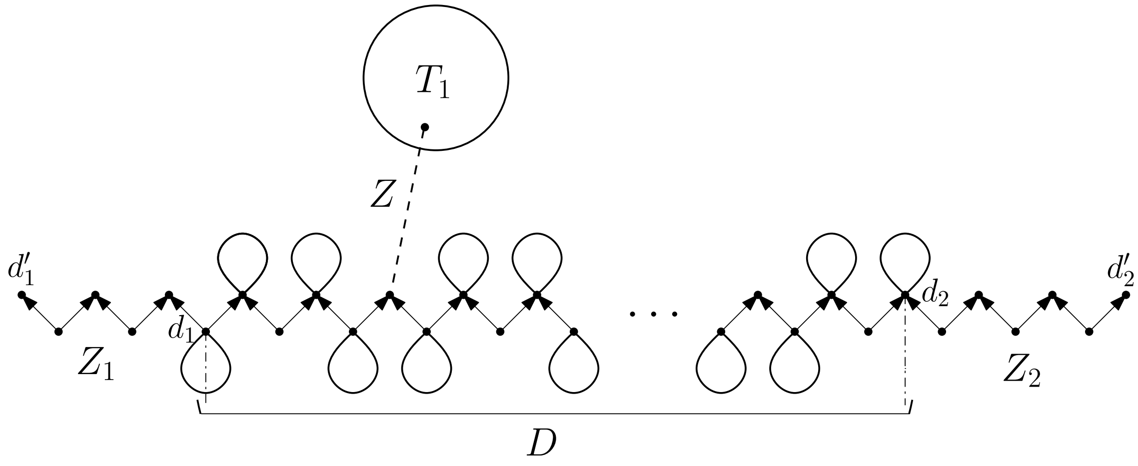

Figure 3.13: This is an example of how I might look.

and length less than $|T_1|/2$ in a way that d_1' and d_2' have the same level. Finally, let I be the tree obtained from the joining of such tree and T_1 by a proper and long enough zig-zag Z as in Lemma 2.1.4. See Figure 3.13. By construction, $T_1 < I < T_2$.

We claim that $I(d_1', d_2')$ satisfies the conditions of Lemma 3.5.1.

Given an oriented path Q, let $\Phi_I(Q)$ be the tree obtained by replacing each arc $v_1v_2 \in A(Q)$ by a copy of I identifying v_1 with d_1' and v_2 with d_2'. Observe that there is a homomorphism

$$h : \Phi_I(Q) \to T_2$$

which maps each vertex of a copy $D \subset \Phi_I(Q)$ to its corresponding vertex in T_2, each vertex of a copy $T_1 \subset \Phi_I(Q)$ to its image via some fixed homomorphism $T_1 \to T_2$, and each vertex of the zig-zags Z, Z_1 and Z_2 to the vertex x, u or w. Hence, $T_1 < \Phi_I(Q) < T_2$ and condition *(i)* holds.

Let us see that a homomorphism $f : D \to \Phi_I(Q)$ can not map a labelled vertex of D to a vertex in Z_1, Z_2 or Z. Consider a vertex $x_i \in D$. Let s be a vertex of $X_i \subset D$ adjacent to x_i and let S be the plank $P(x_i, \{s\}) \subset D$. Suppose $f(x_i)$ is a vertex belonging to a zig-zag. Recall that $f(x_i)$ and x_i must have the same level, thus, $f(s)$ must have the same level of w and u. Consider the homomorphism $h \circ f|_S : S \to T_2$ which maps the vertex s in S to either the vertex w or u in T_2. Note that $(h \circ f|_S)(x_i) = x$. Let $t : T_2 \to T_2$ be a mapping equal to $h \circ f|_S$ for the vertices in S and equal to the identity mapping for the rest of vertices. It is easy to check that t is a homomorphism. Then t is a homomorphism different from the identity since $t(s)$ is equal to u or w contradicting that T_2 is rigid.

Consider a homomorphism $f : D \to \Phi_I(Q)$ and a labelled vertex $v \in D$. As we have seen, $f(v)$ can not belong to a zig-zag, so it must be a vertex of some copy of D in $\Phi_I(Q)$. Let D_c be such copy. It follows by adjacency and by our previous observation that the rest of labelled vertices of D must be mapped also to D_c. Then, since D is a core, the labelled vertices of D and D_c are identified. In particular, $f(d_1)$ and $f(d_2)$ must be the initial and ending labelled vertex of D_c. Thus, condition *(ii)* holds.

Finally, consider a homomorphism $g : I_1^{\epsilon_1} I_2^{\epsilon_2} \to \Phi_I(Q)$, where I_1, I_2 are two copies of I. Suppose that $\epsilon_1 = \epsilon_2 = 1$. Suppose that $g(I_1) \subseteq \Phi_I(q_1 q_1')$ and $g(I_2) \subseteq \Phi_I(q_2 q_2')$ for some arcs $q_1 q_1', q_2 q_2' \in A(Q)$. Let $I_1' = \Phi_I(q_1 q_1')$ and $I_2' = \Phi_I(q_2 q_2')$. Then the vertex $d_2 \in I_1$ is mapped to the vertex $d_2 \in I_1'$ and the vertex $d_1 \in I_2$ is mapped to the vertex $d_1 \in I_2'$. The vertex $d_2 \in I_1$ is joined to the vertex $d_1 \in I_2$ by the path $Z_2 Z_1$. Then it can not happen that $I_1' = I_2'$ since in that case $g(d_2)$ and $g(d_1)$ would be at distance greater than $|T_1|$ but $|Z_2 Z_1| < |T_1|$. The only possibility is that $I_1' I_2'$ is a subtree of $\Phi_I(Q)$, which is equivalent to $q_1' = q_2$. Analogously, it is easy to check that for all $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ condition *(iii)* holds.

It follows by Lemma 3.5.1 that Φ_I is an embedding from $(\mathscr{P}, \leq)$ into the interval $[T_1, T_2]$, and thus, $[T_1, T_2]$ is universal. $\square$

Finally, we show the fractal property for all oriented trees.

Theorem 3.5.3. *Let T_1 and T_2 be two oriented trees satisfying $T_1 < T_2$. If the height of T_2 is greater or equal to 4, then the interval $[T_1, T_2]$ is universal.*

Proof. If T_2 is a proper tree then Theorem 3.5.2 implies the desired result. So suppose that the core of T_2 is a path. Then by Theorem 3.2.1 there exists a tree T such that $T_1 < T < T_2$. If T is a poper tree then the interval $[T_1, T]$ is universal, else if the core of T is a path then by Theorem 3.3.4 the interval $[T, T_2]$ is universal. Thus, in any case, $[T_1, T_2]$ is universal. $\square$

Corollary 3.5.4. *The class of trees of height greater or equal to 4 has the fractal property.*

Chapter 4

Concluding Remarks

The structure of the homomorphism order $(\vec{\mathscr{C}}, \leq)$ is quite clear now. On one hand, we have seen that for every tree T there exists a digraph G_T such that the interval $[G_T, T]$ is a gap, and those are all gaps in the order. One the other hand, we have shown that every interval of the form $[G, H]$ where the core of H contains a cycle is universal, and every interval of trees of height greater or equal to 4 is also universal. It might seem that $(\vec{\mathscr{C}}, \leq)$ has the fractal property; every interval is either a gap or universal. However, we can not forget the existence of the linear order $(\mathcal{L}, \leq)$.

Recall that if F is a balanced digraph then every digraph G homomorphic to F $(G \to F)$ is also balanced. Thus, by Proposition 3.1.6 and 3.1.8, we know which are all the cores of balanced digraphs of height less or equal to 3:

$$[P_0, \vec{P}_3] = \{P_0, \vec{P}_1, \vec{P}_2, \vec{P}_3 = L_0, L_1, L_2, \dots\}$$

which are ordered as

$$P_0 < \vec{P}_1 < \vec{P}_2 < \cdots < L_{k+1} < L_k < L_{k-1} \cdots < L_1 < L_0 = \vec{P}_3.$$

Clearly, the interval $[P_0, \vec{P}_3]$ is not universal since it induces a linear order (there are no incomparable elements). As a consequence, $(\vec{\mathscr{C}}, \leq)$ has not the fractal property as defined in this thesis. But this interval is the only exception, and otherwise every interval is either a gap or universal. The following Theorem summarizes the above results.

Theorem 4.0.1. *The homomorphism order of digraphs $(\vec{\mathscr{C}}, \leq)$ has the fractal property except for the interval $[P_0, \vec{P}_3]$.*

Proof. We want to see that every interval of digraphs $[G_1, G_2]$ such that $[G_1, G_2] \nsubseteq [P_0, \vec{P}_3]$ is either universal or a gap.

Let G_1, G_2 be two cores such that $G_1 < G_2$ and $G_2 \notin [P_0, \vec{P}_3]$. If G_2 contains a cycle then by Theorem 2.2.4 the interval $[G_1, G_2]$ is universal. Otherwise, suppose that G_2 is a tree T of height greater or equal to 4. Then there exists a digraph, in particular, a core G_T such that $[G_T, T]$ is a gap. If $G_1 = G_T$ then $[G_1, G_2]$ is a gap. Else, we must have $G_1 < G_T$. By Theorem 3.5.3 G_T must contain a cycle, otherwise the interval $[G_T, T]$ would be universal and not a gap. Then, G_T contains a cycle and $[G_1, G_T]$ is universal again by Theorem 2.2.4.

All in all, $[G_1, G_2]$ is either universal or a gap $[G_T, T]$. $\square$

Theorem 4.0.1 completely settles the question of the properties of density and fractality of the homomorphism order of the category of directed graphs, which turns out to be slightly more complex than the one for graphs.

A relational structure generalises the notion of a relation and of a graph to more relations and to higher (non-binary) arities. A *type* Δ is a sequence $(\delta_1, \ldots, \delta_k)$ of positive integers. A *relational structure* of type Δ, or Δ-structure, is defined as a pair (V, R) where V is a set and $R = (R_1, \ldots, R_k)$ is a sequence where R_i is a δ_i-nary relation on V. Thus, a relational structure of type $\Delta = (2)$ correspond to a digraph, and a relational structure of type $\Delta = (2, 2)$ would correspond to digraphs with blue-green colored arcs. In this scenario, the concept of homomorphism is also generalise to be a mapping between relational structures that preserves all relations between its elements. In fact, most of the concepts defined in digraphs can be generalised to relational structures, as is the case of a cycle, a tree, etc.

The characterization of the gaps shown by Nešetřil and Tardif in [12] is actually valid for general relational structures. That is, the class of relational structures of a fixed type Δ ordered by the homomorphism order satisfies that, for every Δ-structure T homomorphic to a tree, there exists a Δ-structure G_T such that $[G_T, T]$ is a gap in the order, and all gaps have this form. The next natural step of research is to investigate the fractality property of this more general class of relational structures. We believe that the results we have obtained for directed graphs naturally extends to this more general setting. Moreover, we think that the techniques and ideas we have developed will be useful to pursue this investigation which we leave for future work.

References

[1] P. Erdős. Graph theory and probability. *Canad. J. Math.*, 11:34–38, 1959.

[2] J. Fiala, J. Hubička, Y. Long, and J. Nešetřil. Fractal property of the graph homomorphism order. *Europ. J. Combin.*, 66:101–109, 2017.

[3] R. Fraïssé. *Theory of Relations*, volume 118 of *Studies in Logic and the Foundation of Mathematics*. North-Holland, 1986.

[4] Z. Hedrlín. On universal partly ordered sets and classes. *J. Algebra*, 11:503–509, 1969.

[5] P. Hell and J. Nešetřil. *Graphs and Homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, 2004.

[6] P. Hell and X.Zhu. Homomorphisms to oriented paths. *Discrete Math.*, 132:107–114, 1994.

[7] J. Hubička and J. Nešetřil. Finite paths are universal. *Order*, 22:21–40, 2005.

[8] J. Hubička and J. Nešetřil. Some examples of universal and generic partial orders. In M. Grohe and J. A. Makowsky, editors, *Model Theoretic Methods in Finite Combinatorics*, pages 296–318. American Mathematical Society, 2011.

[9] J.B. Johnston. Universal infinite partially ordered sets. *Proc. Amer. Math. Soc.*, 7:507–514, 1956.

[10] J. Nešetřil. The homomorphism structure of classes of graphs. *Combin. Probab. Comput.*, 17:87–184, 1999.

[11] J. Nešetřil and A. Pultr. On classes of relations and graphs determined by subobjects and factorobjects. *Discrete Math*, 22:287–300, 1978.

[12] J. Nešetřil and C. Tardif. Duality theorems for finite structures (characterising gaps and good characterisations). *J. Combin. Theory Ser. B*, 80:80–97, 2000.

[13] J. Nešetřil and X. Zhu. Path homomorphisms. *Math. Proc. Camb. Phil. Soc.*, 120:207–220, 1996.

[14] P. Oviedo. *Universal Intervals in the Homomorphism Order of Digraphs.* B.S. thesis, Charles University, Prague, 2019.

[15] A. Pultr and V. Trnková. *Combinatorial, Albebraic and Topological Representation of Groups, Semigroups and Categories*, volume 22 of *North-Holland Mathematical Library.* North-Holland, 1980.

[16] E. Welzl. Color-families are dens. *J. Theoret. Comput. Sci.*, 17:29–41, 1982.